CRITICAL POINTS OF LAPLACE EIGENFUNCTIONS ON POLYGONS

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Abstract. We study the critical points of Laplace eigenfunctions on polygonal domains with a focus on the second Neumann eigenfunction. We show that if each convex quadrilaterals has no second Neumann eigenfunction with an interior critical point, then there exists a convex quadrilateral with an unstable critical point. We also show that each critical point of a second-Neumann eigenfunction on a Lip-1 polygon with no orthogonal sides is an acute vertex.

1. Introduction

A second Neumann eigenfunction u of the Laplacian approximates the temperature distribution of an insulated domain for large times. The 'hot spots' conjecture [Rch74] [Kwl85] is the assertion that u does not assume its maximum value in the interior of the domain. The conjecture is false for some non-contractible plane domains [BrdWrn99] [Brd05] but is still believed to be true for convex domains. The conjecture is known to be true when the domains are somewhat elongated, for example, the Lip-1 planar domains of [AtrBrd04]. In [JdgMnd20] [JdgMnd22a] we show that the hot spots conjecture holds true for acute triangles thus resolving Polymath 7 [Polymath].

In the present paper, we extend our study of critical points of eigenfunctions to general polygons and we encounter new phenomena. Note that every planar domain may be approximated by polygonal domains, and hence the weak form of the hot spots conjecture—*some second Neumann eigenfunction has no interior maximum*—for all planar domains would follow from the verification of the strong hot spots conjecture—*every second Neumann eigenfunction has no interior maximum*—for all polygonal domains.

Our general approach to the hot spots conjecture is based on the fact that eigenfunctions and their critical points[1] vary continuously as one varies the domain. Roughly speaking, to show that a second Neumann eigenfunction u_0 on a polygon U_0 has no interior critical points, one constructs a path of polygons P_t and an associated path of eigenfunctions u_t so that the eigenfunction u_1 on P_1 has no interior critical points. If one can show that the putative critical points of each u_t are 'stable' under perturbation, then u_0 also has no interior critical points.

In the case of triangles, we took P_1 to be a right isosceles triangle, and we established enough stability to successfully implement this strategy [JdgMnd20] [JdgMnd22a]. Here we show that the strategy is likely to be more difficult to implement if the polygon has more sides.

Theorem 1.1. *If each convex quadrilateral has no interior critical point, then there exists a convex quadrilateral Q, a second Neumann eigenfunction u on Q, and a nonvertex critical point p of u that is not stable under perturbation.*

By 'stable under perturbation' we mean that if Q_n is a sequence of quadrilaterals that converges to Q and u_n is a sequence of second Neumann eigenfunctions on Q_n that converges to u, then each u_n has a critical point p_n so that p_n converges to p. We conjecture that instability does not hold for triangles.

On the other hand, we are able to successfully apply our strategy for resolving the hot spots conjecture on a large class of polygons.

Theorem 1.2. *Suppose that P_t is a path of polygons such that each P_t has exactly two acute vertices, no two sides of P_t are orthogonal, and P_1 is an obtuse triangle. Then the second Neumann eigenvalue of P_0 is simple, and the set of critical points of each eigenfunction consists of the two acute vertices.*

The class of polygons described in Theorem 1.2 is exactly (up to rigid motion) the class of polygons that have no orthogonal sides and satisfy the Lip-1 condition of [AtrBrd04] (see Proposition 7.7). Thus, Theorem 1.2 provides a non-probabilistic proof of the weak hot spots conjecture for Lip-1 domains. Moreover, in contrast to the result

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[1] With the exception of rectangles, the critical set of a second Neumann eigenfunction on a simply connected polygon is finite [JdgMnd22b].

of [AtrBrd04], we find that not only are there no interior critical points but there are also no critical points on the boundary other than the two acute vertices. Recently, Jonathan Rohleder [Rhl21] announced a non-probabilistic proof of the main result of [AtrBrd04].

We now outline the contents of this paper. In section 2, we use the Bessel expansion of an eigenfunction u to understand the nodal set of Xu near a vertex where X is a constant (resp. rotational) vector field.[2] In particular, we show that whether or not an arc subset of the nodal set of Xu ends at the vertex is essentially determined by the first two Bessel coefficients, the angle at v, and the angle between the vector field and the sides adjacent to v (resp. location of central point). These criteria will be used crucially in the proof of Theorem 1.2.

We will need to rule out the possibility that critical points of a sequence of eigenfunctions, associated to a convergent sequence of polygons, converge to a vertex of the limiting polygon. In §3 we show in various contexts that if critical points converge to a vertex v, then the first two Bessel coefficients of the limiting eigenfunction equal zero. If the limiting polygon is simply connected then this is impossible (Proposition 6.3).

To check the stability of a critical point under perturbation, we will use a variant of the Poincaré-Hopf index. In §4, we define this invariant to include vertices and we prove a variant of the Poincaré-Hopf index formula for Neumann eigenfunctions u on polygons. We relate the index of a critical point of u located at a vertex v with the first two Bessel coefficients of u at v. We also show that the 'total local index' is unchanged under perturbation (Theorem 4.14). As a consequence each non-zero index critical point is stable (Lemma 4.15).

In §5 we provide a local normal form for an eigenfunction in a neighborhood of a critical point p of u whose Poincaré-Hopf index equals zero (Lemma 5.1). Using this local normal form, we find that an index zero critical point cannot be a degree 1 vertex of the nodal set of Xu where X is either a constant or rotational vector field.

In §6 we specialize to simply connected polygons. For such domains, the nodal set of a second Neumann eigenfunction u is a simple arc, and from this fact we deduce that at least one of the first two Bessel coefficients at each vertex is nonzero. This implies a tighter relationship between the index of a vertex critical point of u and the first two Bessel coefficients (Corollary 6.4).

In §7 we prove Theorem 1.2 (Theorem 7.3). We first show that if a polygon P has at least one acute vertex and a second Neumann eigenfunction u on P has an interior critical point, then either u has four non-zero index critical points or there exists a side e of P such that the nodal set of the derivative of u in the direction $\cancel{u}$ of e has an arc that ends at a vertex v of P. This leads us to consider, for the path u_t in Theorem 1.2, the number, $S(t)$, of nonzero index critical points and the number, $V(t)$, of vertices that are endpoints of a nodal arc of the derivative of u_t in the direction of a side of P. We show that the set A of $t \in [0, 1]$ such that either $S(t) \geq 3$ or $V(t) \geq 1$ is open and closed. For the obtuse triangle P_1, we have $S(1) = 2$ and $V(1) = 0$, and hence A is empty. In particular, the initial polygon P_0 has at most two non-zero index critical points, and from this we deduce using the results of §5 that there are no zero index critical points. Using the fact that $V(0) = 0$, we find that the two critical points are located at the vertices of P_0. These two critical points are the unique global extrema, and this implies that the eigenspace is one-dimensional.

In §8 we provide a criterion for the instability of a critical point on a quadrilateral. This criterion is based on the fact that the index of a vertex with angle less than π cannot equal -1 (Corollary 6.4). In particular, an index -1 critical point cannot cross from one side adjacent to a vertex to the other side of the vertex if the angle at the vertex is in $(\pi/2, \pi)$. Hence one is led to find a path of quadrilaterals Q_t such that Q_0 has an index -1 critical point that lies on one side of a vertex and Q_1 and has an index -1 critical point on the other side of the vertex.

In §9 we construct such a path of quadrilaterals and thus prove Theorem 1.1 (Theorem 9.5). The path is constructed by taking a nearly isosceles triangle whose vertex v of smallest angle is less than $\pi/3$, and then 'breaking' the side opposite to v.

In §10 we specialize to convex polygons and find that if a second Neumann eigenfunction has only three critical points then one is a minimum, one is a maximum, and the third has index zero.

2. Solutions to $\Delta u = \lambda u$ on a sector

To understand the behavior of an eigenfunction in a neighborhood of a vertex v of angle β of a polygon, we will consider its Fourier-Bessel expansion. By performing a rigid motion, we may assume that the vertex v is the origin, one side adjacent to v lies in the ray $\{z = r \,:\, r \geq 0\}$ the nonnegative real axis, and the other side lies in the ray $\{z = r \cdot e^{i\beta} \,:\, r \geq 0\}$. If u is a (real) solution to $\Delta u = \lambda u$ with eigenvalue μ that satisfies Neumann conditions on

[2] Here we regard each vector field X as a first order differential operator.

the the rays $\theta = 0$ and $\theta = \beta$, then separation of variables leads to the Fourier-Bessel expansion:

$$u\left(re^{i\theta}\right) \;=\; \sum_{n=0}^{\infty} c_n \cdot J_{\frac{n\pi}{\beta}}(\sqrt{\mu}\cdot r)\cdot \cos\left(\frac{n\pi\theta}{\beta}\right). \tag{1}$$

Here $c_n \in \mathbb{R}$ and J_ν denotes the Bessel function of the first kind of order ν [Lebedev]

$$J_\nu(x) \;=\; x^\nu \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \cdot x^{2k}}{2^{2k}\cdot \Gamma(k+\nu)\cdot \Gamma(k+\nu+1)} \tag{2}$$

where Γ is the Gamma function.

If u satisfies Dirichlet conditions on the rays $\theta = 0$ and $\theta = \beta$, then the one replaces cos with sin, and if u satisfies Dirichlet conditions on the ray $\theta = 0$ and Neumann conditions on the ray $\theta = \beta$, then one replaces $\cos(n\pi\theta/\beta)$ with $\sin(n\pi\theta/2\beta)$ and $J_{n\pi/\beta}$ with $J_{n\pi/2\beta}$.

From (2) we find that, for each $\nu \geq 0$, there exists an entire function g_ν so that $J_\nu(\sqrt{\mu}\cdot r) = r^\nu \cdot g_\nu(r^2)$.[3] Note that neither g_ν nor g'_ν vanishes in a neighborhood of 0 for each $\nu \geq 0$. With this notation, (1) takes a more compact form

$$u\left(re^{i\theta}\right) \;=\; \sum_{n=0}^{\infty} c_n \cdot r^{n\cdot\nu}\cdot g_{n\cdot\nu}\left(r^2\right)\cdot \cos\left(n\cdot\nu\cdot\theta\right) \tag{3}$$

where $\nu = \pi/\beta$. Note that we are suppressing the dependence of g on the eigenvalue μ.

Given a function f, let $\mathcal{Z}(f) = f^{-1}(0)$ denote the *nodal set* of f. For each $\psi \in \mathbb{R}$, let L_ψ denote the constant vector field defined by

$$L_\psi u \;=\; \cos(\psi)\cdot\partial_x \;+\; \sin(\psi)\cdot\partial_y. \tag{4}$$

Lemma 2.1. *Let u be a solution to $\Delta u = \lambda u$ that satisfies Neumann conditions on the rays $\theta = 0$ and $\theta = \beta$.*

(a) If $c_0 \neq 0$ and either $0 < \beta < \pi/2$ or $c_1 = 0$, then there exists an arc in $\mathcal{Z}(L_\psi u)$ with an endpoint at the vertex v if and only if $\psi \in [\pi/2, \pi/2+\beta] \mod \pi$.

(b) If $c_1 \neq 0$ and $\pi/2 < \beta < \pi$, then there exists an arc in $\mathcal{Z}(L_\psi u)$ with an endpoint at the vertex if and only if $\psi \in [\beta - \pi/2, \pi/2] \mod \pi$.

(c) If $c_1 \neq 0$ and $\beta = \pi$, then there exists an arc in $\mathcal{Z}(L_\psi u)$ with an endpoint at the vertex if and only if $\psi = \pi/2 \mod \pi$. Moreover, near the vertex v, this arc lies on ∂P.

(d) If $c_1 \neq 0$ and $\beta > \pi$, then there exists an arc in $\mathcal{Z}(L_\psi u)$ with an endpoint at the vertex if and only if $\psi \in [\pi/2, \beta - \pi/2] \mod \pi$.

Moreover, in all of the above situations, $\mathcal{Z}(L_\psi u)$ has at most one arc with an endpoint at the origin. Figure 1 describes some of these situations.

Proof. If $\psi = \pi/2 \mod \pi$ (resp. $\psi = \beta - \pi/2 \mod \pi$) then because u satisfies Neumann conditions, the arc corresponding to $\theta = 0$ (resp. $\theta = \beta$) lies in ~~both~~ $\mathcal{Z}(L_\psi u)$.

Because $\partial_x = \cos(\theta)\cdot\partial_r - \sin(\theta)r^{-1}\cdot\partial_\theta$ and $\partial_y = \sin(\theta)\cdot\partial_r + \cos(\theta)r^{-1}\cdot\partial_\theta$, we have

$$L_\psi \;=\; \cos(\psi-\theta)\cdot\partial_r \;+\; \sin(\psi-\theta)\cdot\frac{1}{r}\partial_\theta. \tag{5}$$

If $c_0 \neq 0$ and either $c_1 = 0$ or $0 < \beta < \pi/2$, inspection of (3) shows that

$$u\left(re^{i\theta}\right) \;=\; a \;+\; b\cdot r^2 \;+\; f(r,\theta) \tag{6}$$

where a and b are constants and $|L_\psi f| = o(r)$ and $|\partial_\theta L_\psi f| = o(r)$. In particular, we find that

$$L_\psi u\left(re^{i\theta}\right) \;=\; 2b\cdot r\cdot\cos(\psi-\theta) \;+\; o(r), \tag{7}$$

and

$$\partial_\theta L_\psi u\left(re^{i\theta}\right) \;=\; -2b\cdot r\cdot\sin(\psi-\theta) \;+\; o\left(r\right). \tag{8}$$

If $\psi \in (\pi/2, \pi/2+\beta) \mod \pi$, then from (7) we find that $L_\psi u(r)$ and $L_\psi u(re^{i\beta})$ have opposite signs for sufficiently small r. Thus, by the intermediate value theorem, there exists $\gamma(r) \in (0,\beta)$ so that $(L_\psi u)\left(re^{i\gamma(r)}\right) = 0$. Moreover,

[3] ~~Note that though g_ν depends on the eigenvalue μ, we will suppress μ from the notation.~~

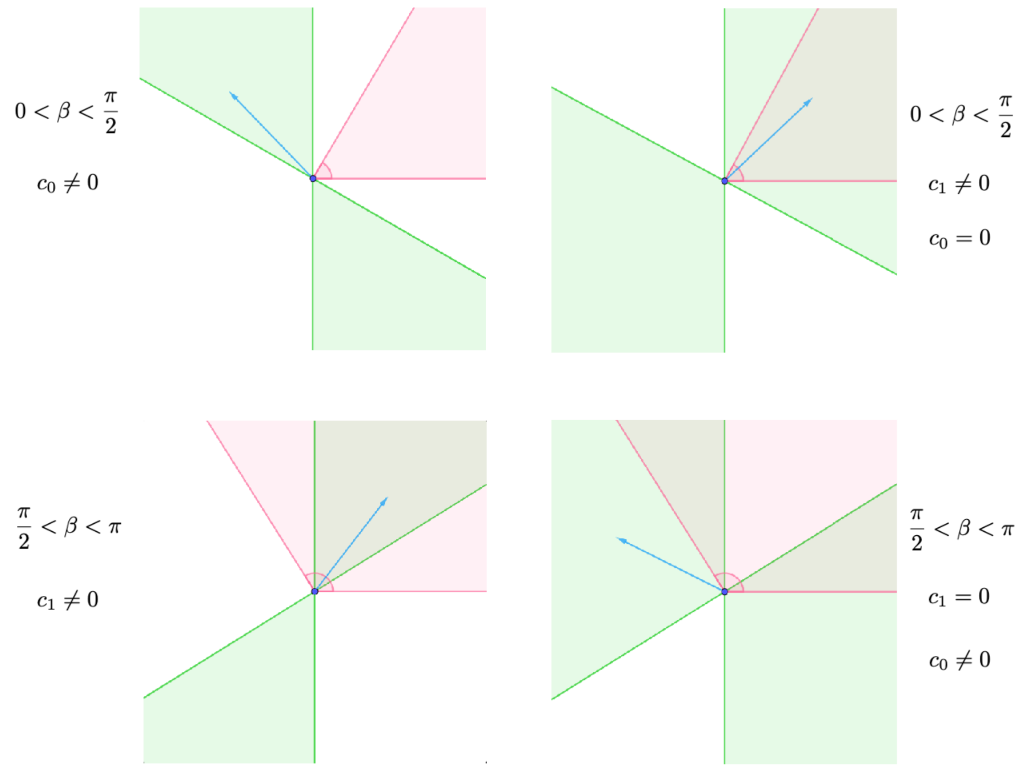

FIGURE 1. Some of the cases described by Lemma 2.1. The region shaded pink is a neighborhood of the vertex v with angle β. The region shaded green describes the directions L_ψ for which $L_\psi u$ has a nodal arc ending at v.

from (8), we find that the point $\gamma(r)$ is unique for sufficiently small r, and by the implicit function theorem, γ is smooth. The map $r \mapsto re^{i\gamma(r)}$ is the desired arc in $\mathcal{Z}(L_\psi u)$. The uniqueness of $\gamma(r)$ implies that there is at most one arc.

Thus we have proven (a). To prove (b), note that if $\beta > \pi/2$ or $c_0 = 0$, then inspection of (3) shows that

$$u\left(r \cdot e^{i\theta}\right) \;=\; c_0 \;+\; c_1 \cdot r^\nu \cdot \cos(\nu \cdot \theta) \;+\; o(r^\nu).$$

and hence a straightforward computation gives

$$L_\psi u\left(r \cdot e^{i\theta}\right) \;=\; c_1 \cdot \nu \cdot r^{\nu-1} \cdot \cos\left(\psi + (\nu - 1) \cdot \theta\right) \;+\; o(r^{\nu-1}).$$

Note that $\cos\left(\psi + (\nu - 1) \cdot \theta\right)$ vanishes if and only if

$$\psi \;=\; \frac{\pi}{2} + (\beta - \pi) \cdot \frac{\theta}{\beta} \mod \pi.$$

One argues as in the proof of part (a) to obtain a unique arc of $\mathcal{Z}(L_\psi u)$ that ends at v.

Parts (c) and (d) follow from arguments similar those that verified (a) and (b).

□

In order to succinctly formulate some corollaries of Lemma 2.1 we make the following definition.

Definition 2.2. Suppose $\beta \neq \pi/2$ and let u be a solution to $\Delta u = \lambda u$ on the sector of angle β that satisfies Neumann conditions. We define the *leading Bessel coefficient* of u at v to be

- c_0 if $\beta < \pi/2$, and
- c_1 if $\beta > \pi/2$.

Corollary 2.3. *If L is a nonzero constant vector field parallel to one the boundary rays of the sector of angle $\beta \neq \pi/2$, then the leading coefficient vanishes if and only if an arc of $\mathcal{Z}(Lu)$ ends at v.*

Proof. Since L is parallel to one of the boundary rays, the vector field L is a multiple of L_ψ with $\psi = 0$ or β. In particular, $\mathcal{Z}(Lu) = \mathcal{Z}(L_\psi u)$. By using the reflection symmetry about $\theta = \beta/2$, we may assume without loss of generality that $\psi = 0$.

Suppose that the leading coefficient does not vanish. If $\beta < \pi/2$, then part (a) of Lemma 2.1 implies that no arc in $\mathcal{Z}(L_\psi u)$ ends at the vertex. Similarly, parts (b), (c), and (d) imply that no arc in $\mathcal{Z}(L_\psi u)$ ends at the vertex in the other cases.

Conversely, suppose that the leading coefficient does equal zero. Let k be the smallest positive integer such that $c_k \neq 0$. If $\beta < \pi/2$, then $c_0 = 0$ and from (3) we find that

$$u\left(r \cdot e^{i\theta}\right) \;=\; r^{k\cdot\nu} \cdot \cos(k \cdot \nu \cdot \theta) \;+\; o\left(r^{k\nu}\right).$$

Hence using (4) we find that

$$L_\psi u\left(r \cdot e^{i\theta}\right) \;=\; k \cdot \nu \cdot r^{k\cdot\nu-1} \cdot \cos\left((k \cdot \nu - 1) \cdot \theta\right) + \; o\left(r^{k\nu-1}\right).$$

Since $\beta < \pi/2$ and $k \geq 1$, the function $\cos((k \cdot \nu - 1) \cdot \theta)$ vanishes for some $\theta \in (0, \beta)$. An implicit function theorem argument establishes the existence of a smooth arc.

If $\beta > \pi/2$, then a similar argument applies to give the claim. □

Corollary 2.4. *Let u be a solution to $\Delta u = \lambda u$ on a sector of angle β not equal to an integral multple of $\pi/2$ satisfying Neumann conditions. If one of the first two Bessel coefficients of u is non-zero and if L is a constant vector field such that*

- *some arc in $\mathcal{Z}(Lu)$ ends at the vertex, and*
- *L is not orthogonal to a boundary ray of the sector,*

then for each constant vector field L' that is sufficiently close to L, some arc of $\mathcal{Z}(L'u)$ ends at the vertex.

Proof. Since β is not equal to an integral multiple of $\pi/2$, by Proposition 4.4 [JdgMnd20], there exists a neighborhood N of the vertex v of the sector that contains no critical points of u.

Since at least one of the zeroth and the first Bessel coefficients of u at v is non-zero, by Lemma 2.1, the set $\mathcal{Z}(Lu)$ contains exactly one arc that ends at v. In particular, u has opposite signs on the the two rays of the sector. By continuity, for each constant vector field L' that is sufficiently close to L, some arc of $\mathcal{Z}(L'u)$ must end at some point $p_{L'}$ near v (depending on L') that lies on the boundary of the sector.

To finish the proof it suffices to show that $p_{L'} = v$ for L' sufficiently close to L. If $p_{L'}$ is not v, then since L is not orthogonal to the sides of the sector, so is L', and hence, $p_{L'}$ is a critical point of u. If L' is sufficiently close then, by continuity, $p_{L'}$ lies in N. This contradicts the first paragraph of the proof. □

Let S_β denote the sector $\{z = r \cdot e^{i\theta} \,:\, \theta \in [0, \beta] \bmod \pi\}$. Recall that R_w denotes the vector field that corresponds to rotation about $w \in \mathbb{C}$.

Corollary 2.5. *Let u be a solution to $\Delta u = \lambda u$ that satisfies Neumann conditions on the sides $\theta = 0$ and $\theta = \beta$. Suppose that c_0 and c_1 are not both equal to zero.*

(1) If $\beta < \pi/2$
   - *(a) and $c_0 \neq 0$, then an arc of $\mathcal{Z}(R_w)$ ends at the vertex if and only if w lies in S_β.*
   - *(b) and $c_0 = 0$, then an arc of $\mathcal{Z}(R_w)$ ends at the vertex if and only if w does not lie in S_β.*

(2) If $\pi/2 < \beta < \pi$
   - *(a) and $c_1 \neq 0$, then an arc of $\mathcal{Z}(R_w)$ ends at the vertex if and only if w does not lie in S_β.*
   - *(b) and $c_1 = 0$, then an arc of $\mathcal{Z}(R_w)$ ends at the vertex if and only if w lies in S_β.*

Proof. If $w = \rho \cdot e^{i\varphi}$, then a computation shows that the rotational vector field about w takes the form

$$R_w \;=\; \partial_\theta \;+\; L_{\varphi+\frac{\pi}{2}}.$$

Because $|\partial_\theta u| = o(r^\nu)$ and $|\partial_\theta^2 u| = o(r^\nu)$, we find that (7) and (8) still hold with L_ψ replaced by R_w and ψ replaced by $\varphi + \pi/2$. Thus the argument given in the proof of Lemma 2.1 applies. □

Remark 2.6. Similar statements hold for Dirichlet and mixed boundary conditions. We leave the formulation of the statements to the reader.

3. Critical points on a sector converging to a vertex

Let S_n be a sequence of sectors that converges to a sector S. Let $u_n : S_n \to \mathbb{R}$ be a sequence of solutions to $\Delta u = \lambda u$ each satisfying Neumann conditions that converges to a Neumann eigenfunction $u : S \to \mathbb{R}$. In this section we show that if certain types of critical points of u_n converge to the vertex of S, then the first two Bessel coefficients of u must vanish. Some of these results are straightforward extensions of results in [JdgMnd20], but several are new.

Let b_- and b_+ denote the distinct boundary rays of the sector S. Let c_0 and c_1 denote the respective Bessel coefficients of u at the vertex of S. Let β denote the vertex angle of S, and let $\nu = \pi/\beta$.

Lemma 3.1 (Compare Proposition 9.1 [JdgMnd20])**.** *For each n, let p_n be a critical point of u_n that lies in the interior of S_n. If p_n converges to the vertex of S, then $c_1 = 0$. If, in addition, $\beta < \pi$, then $c_0 = 0$.*

Proof. Let β_n be the angle of the sector S_n and let $\nu_n = \pi/\beta_n$. By performing rigid motions if necessary, we may assume without loss of generality that the vertex S and each of S_n is 0 and that the boundary rays of S_n are $\theta = 0, \beta_n$. Using (3) and the fact that $\sin(\alpha)$ divides $\sin(k\alpha)$ for each k, we find that

$$\partial_\theta u_n\left(r \cdot e^{i\theta}\right) \;=\; -\nu_n \cdot r^{\nu_n} \cdot \sin(\nu_n \cdot \theta) \cdot \left(c_1(n) \cdot g_{\nu_n}(r) \;+\; O(r^{\nu_n})\right).$$

Thus, since $p_n = r_n \exp(i\theta_n)$ is a critical point, $0 < \theta_n < \beta_n$, and $g_\nu(0) \neq 0$, we find that $c_1(n) = O(r_n^{\nu_n})$. In particular, since u_n converges to u, we have $c_1 = \lim_{n\to\infty} c_1(n) = 0$.

From (3), we find

$$\partial_r u_n\left(r \cdot e^{i\theta}\right) \;=\; c_0(n) \cdot 2r \cdot g_0'(r^2) \;+\; c_1(n) \cdot \nu_n \cdot r^{\nu_n - 1} \cdot g_{\nu_n}(r^2) \cdot \cos(\nu_n\theta) \;+\; O\left(r^{\nu_n}\right).$$

Thus, since $p_n = r_n \exp(i\theta_n)$ is a critical point, $g_0'(0) \neq 0$, and $c_1(n) = O(r^{\nu_n})$, we find that $c_0(n) = O(r^{2(\nu_n - 1)}) + O(r^{\nu_n - 1})$. If $\beta < \pi$, then there exists $\epsilon > 0$ so that for sufficiently large n, we have $\nu_n > 1 + \epsilon$. Hence $c_0 = \lim_{n\to\infty} c_0(n) = 0$. □

Lemma 3.2 (Compare Lemma 9.2 [JdgMnd20])**.** *For each n, let p_n be a critical point of u_n that lies in the boundary ray of S_n that converges to b_-, and let q_n be a critical point of u_n that lies in the boundary ray of S_n that converges to b_+. If the sequences p_n and q_n both converge to the vertex of S, then $c_1 = 0$. If $\beta < \pi$, then we also have $c_0 = 0$.*

Proof. Let β_n be the angle of the sector S_n. By performing rigid motions if necessary, we may assume without loss of generality that the vertex S and each of S_n is 0 and that the boundary rays of S_n are $\theta = 0, \beta_n$. Thus, there exist sequences r_n and s_n so that $p_n = r_n$ and $q_n = s_n e^{i\beta_n}$.

From (3) we find that

$$\begin{aligned}
\partial_r u_n(r) &= c_0(n) \cdot 2r \cdot g_0'(r^2) \;+\; c_1(n) \cdot \nu_n \cdot r^{\nu_n - 1} \cdot g_{\nu_n}(r^2) \;+\; O\left(r^{\nu_n + 1} + r^{2\nu_n - 1}\right) \\
\partial_r u_n\left(s \cdot e^{i\beta_n}\right) &= c_0(n) \cdot 2s \cdot g_0'(s^2) \;-\; c_1(n) \cdot \nu_n \cdot s^{\nu_n - 1} \cdot g_{\nu_n}(s^2) \;+\; O\left(s^{\nu_n + 1} + s^{2\nu_n - 1}\right).
\end{aligned}$$

Since $p_n = r_n$ and $q_n = s_n e^{i\beta_n}$ are critical points, the radial derivative of u_n vanishes at these points, and hence

$$0 \;=\; c_0(n) \cdot 2r_n \cdot g_0'(r_n^2) \;+\; c_1(n) \cdot \nu_n \cdot r_n^{\nu_n - 1} \cdot g_{\nu_n}(r_n^2) \;+\; O\left(r_n^{\nu_n + 1} + r_n^{2\nu_n - 1}\right) \tag{9}$$

$$0 \;=\; c_0(n) \cdot 2s_n \cdot g_0'(s_n^2) \;-\; c_1(n) \cdot \nu_n \cdot s_n^{\nu_n - 1} \cdot g_{\nu_n}(s_n^2) \;+\; O\left(s_n^{\nu_n + 1} + s^{2\nu_n - 1}\right). \tag{10}$$

Let $a_\nu(r) = 2g_0'(r^2)/g_\nu(r^2)$. Because, the functions g_0' and g_ν are continuous and positive near zero, so is a_ν. From (9) and (10) we find that

$$c_0(n) \cdot \left(a_{\nu_n}(r_n) \cdot r_n^{2 - \nu_n} \;+\; a_{\nu_n}(s_n) \cdot s_n^{2 - \nu_n}\right) \;=\; O\left(r_n^2 + s_n^2 \;+\; r_n^{\nu_n} + s_n^{\nu_n}\right), \tag{11}$$

and

$$c_1(n) \cdot \left(\frac{r_n^{\nu_n - 2}}{a_{\nu_n}(r_n)} \;+\; \frac{s_n^{\nu_n - 2}}{a_{\nu_n}(s_n)}\right) \;=\; O\left(r_n^{\nu_n} + s_n^{\nu_n} \;+\; r_n^{2\nu_n - 2} + s_n^{2\nu_n - 2}\right). \tag{12}$$

It follows from (12) that $c_1(n) = O(r_n^2 + s_n^2 + r_n^{\nu_n} + s_n^{\nu_n})$. Since ν_n tends to $\nu > 0$, we have $c_1 = \lim_{n\to\infty} c_1(n) = 0$.

It follows from (11) that $c_0(n) = O(r_n^{\nu_n} + s_n^{\nu_n} + r_n^{2\nu_n - 2} + s_n^{2\nu_n - 2})$. If $\beta < \pi$, then there exists $\epsilon > 0$ so that $\nu_n > 1 + \epsilon$ and for sufficiently large n. Thus, for n sufficiently large, we have $c_0(n) = O(r_n^{1+\epsilon} + s_n^{1+\epsilon} + r_n^{2\epsilon} + s_n^{2\epsilon})$. Therefore, $c_0 = \lim_{n\to\infty} c_0(n) = 0$. □

Lemma 3.3 (Compare Lemma 9.3 [JdgMnd20])**.** *Let p_n be a critical point of u_n and suppose that p_n converges to the vertex of S. If $\beta < \pi/2$, then $c_0 = 0$. If $\beta > \pi/2$, then $c_1 = 0$.*

Proof. By performing rigid motions if necessary, we may assume without loss of generality the boundary rays of S_n are $\theta = 0$ and $\theta = \beta_n$. By Lemma 3.1 passing to a subsequence, and applying a reflection across $\theta = \beta_n/2$ if necessary, we may assume, without loss of generality, that $p_n = r_n$ lies in the positive real axis. As in the proof of Lemma 3.2 we have

$$0 \; = \; c_0(n) \cdot 2r_n \cdot g_0'(r_n^2) \; + \; c_1(n) \cdot \nu_n \cdot r_n^{\nu_n - 1} \cdot g_{\nu_n}(r_n^2) \; + \; O\left(r^{\nu_n+1} + r^{2\nu_n - 1}\right). \tag{13}$$

If $\beta < \pi/2$, then there exists $\epsilon > 0$ so that $\nu_n > 2 + \epsilon$ for n sufficiently large. Hence, since $g_0'(0) \neq 0$, it follows from (13) that $c_0(n) = O(r_n^{\epsilon})$. It follows that $c_0 = 0$.

From (13), we have $c_1(n) \; = \; O(r_n^{2-\nu_n}) + O\left(r_n^2 + r_n^{\nu_n}\right)$. If $\beta > \pi/2$, then there exists $\epsilon > 0$ so that $\epsilon < \nu_n < 2 - \epsilon$ for n sufficiently large. Hence, since $g_\nu(0) \neq 0$, it follows from (13) that $c_1(n) = O(r_n^{\epsilon})$. Thus $c_1 = \lim_{n\to\infty} c_1(n) = 0$. ☐

Lemma 3.4. *Suppose that $\beta \neq \pi/2$ and $\beta < \pi$. Suppose that for each n the sector S_n is bounded by the rays $\theta = 0$ and $\theta = \beta_n$, and there exist $0 < r_n \leq s_n$ such that $\partial_r u(r_n) = 0$ and $\partial_r^2 u(s_n) = 0$. If s_n converges to zero as n tends to infinity, then $c_0 = 0 = c_1$.*

Proof. Because $0 < \beta < \pi$ and $\beta_n \to \beta$, there exists $\delta > 0$ such that $\pi \cdot \delta < \beta_n < \pi \cdot (1+\delta)^{-1}$ and hence $\delta^{-1} > \nu_n > \nu_n - 1 > \delta$. From (3) we have

$$\begin{aligned} \partial_r u_n(r) \; &= \; c_0(n) \cdot 2r \cdot g_0'(r^2) \; + \; c_1(n) \cdot \nu_n \cdot r^{\nu_n - 1} \cdot g_{\nu_n}(r^2) \; + \; O\left(r^{\nu_n+1} + r^{2\nu_n-1}\right) \\ \partial_r^2 u_n(s) \; &= \; c_0(n) \cdot \left(2 \cdot g_0'(s^2) + 4s^2 \cdot g_0''(s^2)\right) \; + \; c_1(n)\nu_n(\nu_n - 1)s^{\nu_n - 1} g_{\nu_n}(s^2) \; + \; O\left(s^{\nu_n} + s^{2\nu_n - 2}\right). \end{aligned} \tag{14}$$

Let

$$a_\nu(r) \; = \; \frac{2g_0'(r^2)}{\nu \cdot g_\nu(r^2)}$$

and

$$b_\nu(s) \; = \; \frac{2g_0'(s^2) + 4s^2 \cdot g_0''(s^2)}{\nu \cdot g_\nu(s^2)}.$$

Because g_ν and its derivatives are positive and continuous for r near zero, the functions a_ν and b_ν are also positive and continuous for small r. Note that $a_\nu(0)/b_\nu(0) = 1$.

Since $\partial_r u(r_n) = 0$ and $\partial_r^2 u(s_n) = 0$ we find from (14) that

$$\begin{aligned} 0 \; &= \; c_0(n) \cdot a_{\nu_n}(r_n) \cdot r_n^{2-\nu_n} \; + \; c_1(n) \; + \; O\left(r_n^2 + r_n^{\nu_n}\right) \\ 0 \; &= \; c_0(n) \cdot \frac{b_{\nu_n}(s_n)}{\nu_n - 1} \cdot s_n^{2-\nu_n} \; + \; c_1(n) \; + \; O\left(s_n^2 + s_n^{\nu_n}\right). \end{aligned} \tag{15}$$

By subtracting we have

$$c_0(n) \cdot \left(a_{\nu_n}(r^{\nu_n}) \cdot r^{2-\nu_n} \; - \; \frac{b_{\nu_n}(s_n)}{\nu_n - 1} \cdot s_n^{2-\nu_n} \right) \; = \; O\left(r_n^2 + r_n^{\nu_n} + s_n^2 + s_n^{\nu_n}\right) \tag{16}$$

Suppose $\beta > \pi/2$, then $\nu < 2$ and so since $\nu_n \to \nu$ there exists $\epsilon > 0$ so that for sufficiently large n

$$\frac{1}{\nu_n - 1} \cdot \frac{b_{\nu_n}(s_n)}{a_{\nu_n}(r_n)} \; \geq \; 1 + \epsilon. \tag{17}$$

Since $r_n \leq s_n$, we have $r_n^{2-\nu_n} \leq s_n^{2-\nu_n}$. Therefore, from (16) we find that

$$c_0(n) \cdot (-\epsilon) \cdot a(r_n) \cdot s^{2-\nu_n} \; = \; O\left(r_n^2 + r_n^{\nu_n} + s_n^2 + s_n^{\nu_n}\right)$$

Thus, since $r_n \leq s_n$ we find that $c_0(n) = O(r_n^{\nu_n} + s_n^{\nu_n} + r_n^{2\nu_n - 2} + s_n^{2\nu_n - 2})$, and hence

$$c_0(n) \; = \; O\left(r_n^{1+\delta} + s_n^{1+\delta} + r_n^{2\delta} + s_n^{2\delta}\right). \tag{18}$$

Therefore, $c_0 = \lim_{n\to\infty} c_0(n) = 0$.

Suppose $\beta < \pi/2$. Then since $\nu_n \to \nu > 2$, there exists $\epsilon > 0$ so that for sufficiently large n

$$(\nu - 1) \cdot \frac{a_{\nu_n}(r_n)}{b_{\nu_n}(s_n)} \; \geq \; 1 + \epsilon. \tag{19}$$

Since $r_n^{2-\nu_n} \geq s^{2-\nu_n}$, from (16) one deduces that $c_0 = 0$ in this case by arguing in a similar manner.

To show that $c_1 = 0$, we argue similarly. From (15) we find that

$$0 \;=\; c_0(n) \;+\; c_1(n) \cdot \frac{r_n^{\nu_n - 2}}{a_{\nu_n}(r_n)} \;+\; O\left(r_n^2 + r_n^{\nu_n}\right)$$

$$0 \;=\; c_0(n) \;+\; c_1(n) \cdot \frac{(\nu_n - 1) \cdot s_n^{\nu_n - 2}}{b_{\nu_n}(s_n)} \;+\; O\left(s_n^2 + s_n^{\nu_n}\right).$$

and hence by subtracting

$$c_1(n) \cdot \left(\frac{r_n^{\nu_n - 2}}{a_{\nu_n}(r_n)} \;-\; \frac{(\nu_n - 1) \cdot s_n^{\nu_n - 2}}{b_{\nu_n}(s_n)} \right) = \; O\left(r_n^2 + r_n^{\nu_n} + s_n^2 + s_n^{\nu_n}\right).$$

Now argue as was done to show that $c_0 = 0$. In particular, in the case $\beta < \pi/2$ use (19), and in the case $\beta > \pi/2$ use (17).

□

Corollary 3.5. *Suppose $\beta < \pi$ and $\beta \neq \pi/2$. Suppose that for each n, the points p_n and q_n are distinct critical points. If p_n and q_n both converge to the vertex of S, then $c_0 = 0 = c_1$.*

Proof. By applying rigid motions we may assume that S_n is bounded by the rays $\theta = 0$ and $\theta = \beta_n$. By Lemma 3.1 and Lemma 3.2, it suffices to assume that p_n and q_n lie in the same boundary ray, and by reflecting if necessary about $\theta = \beta_n/2$, we may assume that both p_n and q_n are real. By relabeling we may assume that $p_n < q_n$. By assumption $\partial_r(p_n) = 0 = \partial_r(q_n)$, and so Rolle's theorem implies that there exist s_n such that $p_n \leq s_n \leq q_n$ and $\partial_r^2(s_n) = 0$. The claim now follows from Lemma 3.4. □

Corollary 3.6. *Let S be a sector with angle $\beta < \pi$ and not equal to $\pi/2$, and let $u : S \to \mathbb{R}$ be a Neumann eigenfunction. If the vertex v is an accumulation point of the critical points of u, then $c_0 = 0 = c_1$.*

Proof. Apply Corollary 3.5 with $S_n = S$ and $u_n = u$. □

Lemma 3.7. *Suppose $\beta < \pi$ and $\beta \neq \pi/2$. If p_n is a degenerate critical point of u_n that converges to the vertex of S, then $c_0 = 0 = c_1$.*

Proof. By applying rigid motions we may assume that S_n is bounded by $\theta = 0$ and $\theta = \beta_n$. By Lemma 3.1, by passing to a subsequence, and by applying a reflection across $\theta = \beta_n/2$ if necessary, we may assume that p_n lies in the boundary ray $\theta = 0$. That is, $p_n = r_n > 0$ and $\partial_r u_n(r_n) = 0$.

Since u_n satisfies Neumann conditions along the real axis, and p_n is a degenerate critical point we have either $\partial_x^2 u_n(r_n) = 0$ or $\partial_y^2 u_n(p_n) = 0$. If $\partial_x^2 u_n(r_n) = 0$, then Lemma 3.4 with $s_n = r_n$ implies the claim.

Suppose then that $\partial_y^2 u_n(p_n) = 0$. Along the ray $\theta = 0$ we have $\partial_y^2 = r^{-1} \cdot \partial_r + r^{-2} \cdot \partial_\theta^2$. Since $\partial_r u_n(r_n) = 0$, we have $\partial_y^2 u_n(r_n) = \partial_\theta^2 u_n(r_n)$, and so

$$0 \;=\; \left(\partial_\theta^2 u_n\right)(r_n) \;=\; -c_1(n) \cdot \nu_n^2 \cdot r^{\nu_n} \cdot g_0(r_n^2) \;+\; O(r_n^{2\nu_n}).$$

Since u_n satisfies the first equation in (13) we find that

$$0 \;=\; \left(\partial_y^2 u_n\right)(r_n) \;=\; 2c_0(n) \cdot g_0'(r_n^2) \;+\; O\left(r_n^{\nu_n} + r_n^{2\nu_n - 2}\right).$$

Since $\beta < \pi$, there exists $\epsilon > 0$ so that $\nu_n > 1 + \epsilon$ for suficiently large n. Since g_0 and its derivative do not vanish at zero, it follows that $c_0 = \lim_{n\to\infty} c_0(n) = 0$ and $c_1 = \lim_{n\to\infty} c_1(n) = 0$.

□

Remark 3.8. Note that in the proof of Lemma 3.7 we used the condition $\beta \neq \pi/2$ only in the case that $\partial_r^2 u_n(p_n) = 0$. Indeed, the proof shows that if $\partial_\theta^2 u_n(p_n) = 0$, then $c_0 = 0 = c_1$ even if $\pi = \beta/2$.

4. A Poincaré-Hopf formula for critical points of eigenfunctions on a polygon

In this section, we provide a variant of the the classical-Poincaré Hopf index theorem for the gradient of Laplace eigenfunctions on a planar polygonal domain P. The discussion will focus on eigenfunctions satisfying Neumann boundary conditions, but the methods apply to give variants in the contexts of Dirichlet and mixed boundary conditions.

Each Neumann eigenfunction $u : P \to \mathbb{R}$ extends continuously to the boundary ∂P, and this extension is smooth at each nonvertex point in ∂P. Let p lie in the closure $\overline{P}$ of P. Suppose that there exists a deleted disc neighborhood

$\dot{D}$ of p that contains no zeros of ∇u. Then the closure of each component of $\dot{D} \cap \{z : u(z) = u(p)\}$ is an arc.[4] If such an arc contains p, then we will say that the arc *emanates* from p. Let n be the number of arcs in $\{z : u(z) = u(p)\}$ that emanate from p, and define

$$\mathrm{ind}(u,p) \;=\; \begin{cases} 1 \;-\; \frac{1}{2}\cdot n & \text{if } p \in P \\ \;\;1 \;-\; n & \text{if } p \in \partial P. \end{cases}$$

Note that if $\mathrm{ind}(u,p) \neq 0$ and p is not a vertex of P, then $\nabla u(p) = 0$.[5] If p is a vertex and $\mathrm{ind}(u,p) \neq 0$, then we will regard p as a critical point of u.

Definition 4.1. A point $p \in \overline{P}$ will be called a *critical point* of u if either

- p is not a vertex and $\nabla u(p) = 0$, or
- p is a vertex and $\mathrm{ind}(u,p) \neq 0$.

Assumption 4.2. *In what follows we will assume that each critical point p is isolated. In particular, index* $\mathrm{ind}(u,p)$ *is well-defined for each p.*

In [JdgMnd22b], we show that rectangles are the only simply-connected polygons whose second Neumann eigenfunctions have infinitely many critical points. Hence the assumption reduces to the assumption that the polygon is not a rectangle in the simply-connected case.

Let $\chi(S)$ denote the Euler characteristic of a surface S.[6] For example, if S is a polygonal domain obtained by removing k disjoint simply connected polygons from the interior of a simply connected polygon, then $\chi(S) = 1-k$. Let $\mathrm{crit}(u)$ denote the set of critical points of u including the vertices v such that $\mathrm{ind}(u,v) \neq 0$. The following is a variant of the classical Poincaré-Hopf formula [Taylor].

Proposition 4.3 (Index formula)**.** *Let $u : P \to \mathbb{R}$ be a Neumann eigenfunction such that the set* $\mathrm{crit}(u)$ *is finite.*

$$2\cdot\chi(P) \;=\; \sum_{p\,\in\,\mathrm{crit}(u)\cap P} 2\cdot\mathrm{ind}(u,p) \;+\; \sum_{p\,\in\,\mathrm{crit}(u)\cap\partial P} \mathrm{ind}(u,p).$$

Proof. Let DP be the 'double of P', the closed surface without boundary obtained by gluing two disjoint copies of $\overline{P}$ along their respective boundaries. The surface DP has a natural real-analytic structure on the complement of the set C of 'cone points' corresponding to the vertices of P. Because u is a Neumann eigefunction, u extends to a real-analytic function $\widetilde{u} : DP \setminus C \to \mathbb{R}$ that is invariant under the isometric involution that exchanges the two copies of P. For each $p \in DP$, we define $\mathrm{ind}(\widetilde{u},p) = 1 - \frac{n}{2}$. Because u is a Neumann eigenfunction, we find that $\mathrm{ind}(\widetilde{u},p) = \mathrm{ind}(u,p)$ for $p \in P$ (including vertices).

Let A be the union of the level sets of $\widetilde{u}$ that contain critical points of $\widetilde{u}$. The complement of A consists of topological annuli, and hence, by the Euler-Poincaré formula, $\chi(DP) = \chi(A)$. On the other hand, the number of edges in A equals $\frac{1}{2}\sum_p n_p$ where n_p is the valence of the graph A at p. It follows that $\chi(DP) = \sum_{\mathrm{crit}(\widetilde{u})} \mathrm{ind}(u,p)$ where $\mathrm{crit}(\widetilde{u})$ includes $p \in C$ such that $\mathrm{ind}(\widetilde{u},p) \neq 0$. We have $\chi(DP) = 2\cdot\chi(P)$ and for every interior critical point of u we have two critical points of $\widetilde{u}$ with the same index. The claimed formula follows. ☐

Remark 4.4. There are also variants of Proposition 4.3 in the contexts of Dirichlet and mixed boundary conditions. For example, if u satisfies Dirichlet conditions, then formula (4.3) holds true if one redefines $\mathrm{ind}(u,p) = 2-k$ for each for $p \in \partial P$.

Remark 4.5. The classical Poincaré-Hopf theorem applies to a smooth vector field X on an oriented closed surface S that has finitely many critical points. If γ is a simple oriented loop that encloses at most one zero p of X, then the restriction of $X/|X|$ to γ defines a map from the unit circle to itself. The index of X at p is the degree of this self-map of the circle. (See, for example, [Taylor] §1.10.) If $X = \nabla f$, then this index equals $1 - \frac{k}{2}$ where k is the number of components of $f^{-1}(f(p)) \setminus \{p\}$. In the context of a vector field X, the Poincaré-Hopf index formula gives that the sum of the indices of the zeros of X equals the Euler characteristic of S.

Proposition 4.6. *The point $p \in P$ is a local extremum if and only if* $\mathrm{ind}(u,p) = 1$.

[4] If the closure of some component were a loop, then the loop would bound a disk that contained a critical point. In this paper we use 'arc' to mean an embedded interval.

[5] The converse is not true, namely there may be critical points with index equal to zero. See §5.

[6] See for example [Taylor] or [Thurston].

Proof. By Assumption 4.2, each critical point is isolated. We have $\mathrm{ind}(u,p) = 1$ if and only if there exists a punctured disc neighborhood $\dot{D}$ of p so that $u(z) \neq u(p)$ for each $z \in \dot{D}$. Since u is continuous, we have either $u(z) > u(p)$ for all $z \in \dot{D}$ or $u(z) < u(p)$ for all $z \in \dot{D}$. This occurs if and only if p is a local extremum of u. □

Suppose that v is a vertex of P that is not a limit point of the zeros of ∇u. The index $\mathrm{ind}(u,v)$ is determined by the Bessel expansion (3) of u near v.

Lemma 4.7. *Let P be a polygon, let v be a vertex of P with angle β, and let u be a Neumann eigenfunction on P. Let $k \geq 1$ be the smallest positive integer so that $c_k \neq 0$ and suppose that v is a critical point of u.*

(i) If $u(v) = 0$ or $\beta > k \cdot \frac{\pi}{2}$, then $\mathrm{ind}(u,v) = 1-k$.
(ii) If $u(v) \neq 0$ and $\beta < k \cdot \frac{\pi}{2}$, then $\mathrm{ind}(u,v) = 1$.
(iii) If $u(v) \neq 0$ and $\beta = k \cdot \frac{\pi}{2}$, then $1-k \leq \mathrm{ind}(u,v) \leq 1$.

In particular, if $\beta \neq \pi/2$ or $3\pi/2$, then $\mathrm{ind}(u,v)$ equals either 1 or $1-k$.

A similar statement can be derived in the cases of Dirichlet or mixed boundary conditions.

Proof. Without loss of generality, $v = 0$ and the sides adjacent to v bound the sector $0 < \theta < \beta$.

If $u(0) = 0$ or $\beta > k \cdot \frac{\pi}{2}$, then from (3) there exist $b \neq 0$ and a so that

$$u\left(r \cdot e^{i\theta}\right) \;=\; a \;+\; b \cdot r^{k\nu} \cdot \cos(k\nu\theta) \;+\; o\left(r^{k\nu}\right).$$

Using, for example, the implicit function theorem, one finds that there exists a disk neighborhood D of 0 such that $D \cap u^{-1}(u(v)) \setminus \{v\}$ consists of k arcs each with an endpoint at v. It follows that $\mathrm{ind}(u,0) = 1-k$.

Suppose $u(v) \neq 0$ and $\beta < k \cdot \frac{\pi}{2}$. Then $k \cdot \nu > 2$ and hence from (3) we find that

$$u(z) \;=\; a \;+\; b \cdot r^2 \;+\; o(r^2)$$

where $a \neq 0 \neq b$. Hence v is a local extremum of u, and so by Proposition 4.6, $\mathrm{ind}(u,v) = 1$.

If $u(v) \neq 0$ and $\beta = k\pi/2$, then from (3) we have

$$u\left(r \cdot e^{i\theta}\right) \;=\; a \;+\; r^2\left(b \;+ c \cdot \cos(k\nu\theta)\right) \;+\; o\left(r^2\right)$$

where a, b and c are nonzero constants. If $b = -c$, then $\mathrm{ind}(u,v)$ will depend on the $o(r^2)$ error term. In this case $1-k \leq \mathrm{ind}(u,v) \leq 1$. □

Corollary 4.8. *Suppose β is not a multiple of $\pi/2$.*

(1) If $c_1 = 0$, then $\mathrm{ind}(u,v) \neq 0$.
(2) If $\beta > \pi/2$, then $c_1 = 0$ if and only if $\mathrm{ind}(u,v) \neq 0$.

Proof. Let k be as in the statement of Lemma 4.7. If $c_1 = 0$, then $k > 1$, and hence Lemma 4.7 implies that $\mathrm{ind}(u,v) \neq 0$. If $\beta > \pi/2$ and $c_1 \neq 0$, then $k = 1$ and we are in case (i) of Lemma 4.7. Thus, $\mathrm{ind}(u,v) = 0$. □

Remark 4.9. In the case that β is a multiple of $\pi/2$ and $u(v) \neq 0$, part (iii) of Lemma 4.7 provides only an inequality for $\mathrm{ind}(u,v)$. Yet, one can determine the index in finitely many steps. In particular if $k \cdot \nu = 2$, then

$$u(z) - u(v) \;=\; r^2 \cdot (a \;+\; \cos(2\theta)) \;+\; o(r^2)$$

where $a = (c_0 \cdot g_0'(0))/(c_k \cdot g_2(0))$. If $|a| > 1$, then $\mathrm{ind}(u,v) = 1$ and if $|a| < 1$, then $\mathrm{ind}(u,v) = 1-k$. If $|a| = 1$, then by considering more terms of the Bessel expansion, one can identify $\mathrm{ind}(u,v)$.

If p is an isolated critical point of an eignfunction u that lies in the interior of a polygon P, then $\mathrm{ind}(u,p)$ equals the degree of the mapping $\nabla u/|\nabla u| \circ \gamma$ as described in Remark 4.5. If p lies in the interior of a side of P, then one may reflect a Neumann eigenfunction across the side to $\widetilde{u}$, and then find that $\mathrm{ind}(u,p)$ equals degree of the map $\nabla\widetilde{u}/|\nabla\widetilde{u}| \circ \gamma$.

If p is a vertex, we may also interpret $\mathrm{ind}(u,v)$ in terms of the degree of the self-map of the circle induced by a vector field. Indeed, let D be a disc centered at p that intersects no sides of P other than the side(s) adjacent to p and so that $\overline{D} \setminus \{p\}$ contains no critical points of u other than possibly p. By applying a rigid motion we way assume that $p = 0$ and that $D \cap P$ lies in the sector S bounded by the rays $\theta = 0$ and $\theta = \beta$. Moreover, by rescaling if necessary, we may assume that D is the unit disk. The map $z \mapsto z^{\frac{\beta}{\pi}}$ maps $H = \{z \in \mathbb{C} : |z| < 1 \text{ and } y > 0\}$ to the sector $D \cap P$. In particular, the function $w(z) = u\left(z^{\frac{1}{\nu}}\right)$ is defined on H. If u is given by (3), then

$$w\left(r \cdot e^{i\theta}\right) \;=\; \sum_{j=0} c_j \cdot r^j \cdot g_{j \cdot \nu}\left(r^{\frac{2}{\nu}}\right) \cdot \cos(j \cdot \theta). \tag{20}$$

We may extend w smoothly to $D \setminus \{0\}$ by setting $w(\overline{z}) = w(z)$.

Lemma 4.10. *The degree of the restriction of $\frac{\nabla w}{|\nabla w|}$ to the unit circle equals $2 \cdot \sum \operatorname{ind}(u, q)$ where the sum is over critical points q of u that lie D.*

Proof. Suppose $q \neq 0$ is a critical point of u. If q lies in the interior of P, then q corresponds to two critical points q_+ and q_- of w which have the same indices as q. By Remark 4.5, since w is smooth at $q_\pm$, the index $\operatorname{ind}(w, q_\pm)$ equals the degree of the restriction of $\nabla w/|\nabla w|$ to a small circle centered at $q_\pm$. If $q \neq 0$ lies on the boundary of P, then q corresponds to a single critical point q' of w, and $\operatorname{ind}(w, q')$ equals the degree of $\nabla w/|\nabla w|$ on a small circle centered at q'. By choosing disjoint circles, and applying a standard argument[7], we find that it suffices to assume that u has no critical points in $\overline{D} \setminus \{0\}$.

Since u has no critical points in $\overline{D} \setminus \{0\}$, the function w has no critical points in $\overline{D} \setminus \{0\}$ In particular, since $\partial_\theta w$ vanishes on the real line, it follows that $w(z) \neq w(0)$ for each $z \neq 0$ on the real axis. Hence the number of arcs in $\{z : w(z) = w(0)\}$ that emanate from 0 equals twice the number of arcs in $\{z : u(z) = u(0)\}$ that emanate from 0. Thus, to complete the proof, it suffices to show that the degree of $\frac{\nabla w}{|\nabla w|} \circ \gamma$ where γ is the unit circle equals $1 - n/2$ where n is the number of arcs of $\{z : w(z) = w(0)\}$ that emanate from 0.

Let $h(z) := w(z) - w(0)$ and let k be the smallest positive interger such that $c_k \neq 0$. Then

$$\partial_\theta h\left(r \cdot e^{i\theta}\right) \;=\; -c_k \cdot k \cdot r^{k-1} \cdot g_{k \cdot \nu}\left(r^{\frac{2}{\nu}}\right) \cdot \sin(k \cdot \theta) \;+\; O\left(r^k\right),$$

and so there exists $r_0 > 0$ so that if $0 < r \leq r_0$, then the set $\{\theta : \partial_\theta h\left(r \cdot e^{i\theta}\right) = 0\}$ consists of exactly $2k$ elements, $\theta_0(r), \ldots, \theta_{2k-1}(r)$. Using the implicit function theorem, we find that, for each j, the map $r \mapsto \theta_j(r)$ is smooth. By relabeling if necessary, we may assume that $\lim_{r \to 0} \theta_j(r) = j \cdot \pi/k$. The function h has no critical points in $\overline{D} \setminus \{0\}$, and so the degree of $\nabla h/|\nabla h| \circ \gamma$ equals the degree of the map $\nabla h/|\nabla h| \circ \gamma_0$ where γ_0 is the standard counterclockwise parameterization of $r = r_0$.

Choose a homeomorphism $\psi : \overline{D} \to \overline{D}$ that is isotopic to the identity map, that is smooth away from 0, and that maps each ray $\theta = j \cdot \pi/k$ to the arc θ_j. Then if we define $\widetilde{h}(z) = h \circ \psi$, then the degree of $\nabla\widetilde{h}/|\nabla\widetilde{h}| \circ \gamma_0$ equals the degree of $\nabla h/|\nabla h| \circ \gamma_0$ and $\operatorname{ind}(\widetilde{h}, 0) = \operatorname{ind}(h, 0)$.

Let $j \in \{1, \ldots, 2k\}$ and let $\theta_j := j\pi/k$. Since $\widetilde{h}$ has no critical points in $\overline{D} \setminus \{0\}$, the mean value theorem implies that $r \mapsto |\widetilde{h}\left(re^{i \cdot \theta_j}\right)|$ is strictly increasing and thus $\widetilde{h}\left(re^{i \cdot \theta_j}\right) \neq 0$ for each $r \in (0, r_0]$. Let $\epsilon_j \in \{+1, -1\}$ denote the sign of the function $r \mapsto \widetilde{h}\left(re^{i \cdot \theta_j}\right)$. Note that ϵ_j is also the sign of $\partial_r \widetilde{h}\left(re^{i \cdot \theta_j}\right)$.

The number arcs in $\{z : \widetilde{h}(z) = 0\}$ emanating from 0 equals the number of $j \in \{1, \ldots, 2k\}$ such that $\epsilon_j \neq \epsilon_{j+1}$. Indeed, for each fixed r, the restriction of $\theta \mapsto \widetilde{h}\left(re^{i\theta}\right)$ to the interval $I_j := [\theta_j, \theta_{j+1}]$ is monotone, and hence $\theta \mapsto \widetilde{h}\left(re^{i\theta}\right)$ assumes the value 0 at most once, and it assumes the value 0 if and only if $\epsilon_j \neq \epsilon_{j+1}$. In other words, $\operatorname{ind}(\widetilde{h}, 0)$ equals the number of j such that $\epsilon_j \neq \epsilon_{j+1}$.

To compute the degree of $\nabla\widetilde{h}/|\nabla\widetilde{h}| \circ \gamma_0$, we first regard this map as a map $X : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$. In particular, for each θ there exists a unique $X(\theta)$ so that $\nabla\widetilde{h}/|\nabla\widetilde{h}|(r \cdot e^{i\theta})$ corresponds to the point $e^{iX(\theta)}$ in the unit circle. In other words, $X(\theta)$ is the angle between the vector ∂_x and $\nabla\widetilde{h}$ measured counterclockwise.

We have $\nabla\widetilde{h} = \partial_r \widetilde{h} \cdot \partial_r + r^{-2} \cdot \partial_\theta \widetilde{h} \cdot \partial_\theta$. Since $\partial_\theta \widetilde{h}(re^{\theta_j}) = 0$ we have $\nabla\widetilde{h} = \partial_r \widetilde{h} \cdot \partial_r$, and so

$$X(\theta_j) \;=\; \begin{cases} \;\;\theta_j \mod 2\pi & \text{if } \epsilon_j = +1, \\ \theta_j + \pi \mod 2\pi & \text{if } \epsilon_j = -1. \end{cases}$$

We also have $\partial_\theta \widetilde{h}(r_0 e^{i\theta}) > 0$ if and only if $X(\theta) \in (\theta, \theta + \pi) \mod 2\pi$, and $\partial_\theta \widetilde{h}(r_0 e^{i\theta}) < 0$ if and only if $X(\theta) \in (\theta - \pi, \theta)$ In particular, we have either $X(\theta) \in [\theta, \theta + \pi]$ for each $\theta \in I_j$ or $X(\theta) \in [\theta - \pi, \theta]$ for each $\theta \in I_j$.

If $\epsilon_j = +1 = \epsilon_{j+1}$, then $X(\theta_j) = \theta_j$ and $X(\theta_{j+1}) = \theta_{j+1}$ and either $\theta \leq X(\theta) \leq \theta + \pi$ for each $\theta \in I_j$ or $\theta - \pi \leq X(\theta) \leq \theta$ for each $\theta \in I_j$. It follows that the restriction of X to I_j is homotopic to the identity map

[7] See for example, the proof of Proposition 20.2 in [Taylor]

rel endpoints. Similarly, if $\epsilon_j = -1 = \epsilon_{j+1}$, then the restriction of X to I_j is homotopic to the identity map rel endpoints.

If $\epsilon_j = -1$ and $\epsilon_{j+1} = +1$, then $\partial_\theta \widetilde{h}(r_0 e^{i\theta}) \geq 0$ for each $\theta \in I_j$, and so $X(\theta) \in [\theta, \theta + \pi] \mod 2\pi$. We also have $X(\theta_j) = \theta_j + \pi \mod 2\pi$ and $X(\theta_{j+1}) = \theta_{j+1} \mod 2\pi$. It follows that X is homotopic rel endpoints to the linear map $Y_j^+ : I_j \to \mathbb{R}/2\pi\mathbb{Z}$ defined by

$$Y_j^+(\theta) \ = \ (1-k)\cdot(\theta - \theta_j) + \ \theta_j \ + \ \pi \mod 2\pi.$$

Similarly, if $\epsilon_j = +1$ and $\epsilon_{j+1} = -1$, then one finds that $X|_{I_j}$ is homotopic rel endpoints to the map $Y_j^- : I_j \to \mathbb{R}/2\pi\mathbb{Z}$ defined by

$$Y_j^-(\theta) \ = \ (1-k)\cdot(\theta - \theta_j) + \ \theta_j \mod 2\pi.$$

Using the identity map on I_j when $\epsilon_j = \epsilon_{j+1}$ and the maps Y_j^+ and Y_j^- when $\epsilon_j \neq \epsilon_{j+1}$, one constructs a piecewise linear map $Y : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}/2\pi\mathbb{Z}$ that is homotopic to X. An elementary argument shows that Y is in turn homotopic to the map Z defined by $Z(\theta) = (1 - \frac{n}{2})\cdot\theta \mod 2\pi$ where n is the number of j such that $\epsilon_j \neq \epsilon_{j+1}$. The claim follows. ☐

Remark 4.11. If the angle β at v is not a multiple of $\pi/2$, then $k \cdot \nu \neq 2$, and the proof of Lemma 4.10 can be significantly shortened. Indeed, one can use expansion (20) as in the proof of Lemma 4.7. However, if $\beta = \pi/2$ or $3\pi/2$, then using expansion (20) is more cumbersome. See Remark 4.9.

Definition 4.12. Let P_k be a sequence of n-gons and let P be an n-gon. We will say that P_k converges to P if and only if there exists a sequence of homeomorphisms $\varphi_k : \overline{P} \to \overline{P}_k$ that are C^2 diffeomorphisms on the complement of the vertices such that φ_k converges uniformly in C^2 to the identity map on each compact subset of $\overline{P}$ that does not include the vertices. Given continuous functions $u_k : P_k \to \mathbb{C}$ and $u : P \to \mathbb{C}$, we will say that u_k converges to U if and only if $u_k \circ \varphi_k$ converges to u.

Remark 4.13. Our notion of convergence requires that the number, n, of vertices be constant. On the other hand, we will sometimes want to analyze a sequence P_k of polygons with n vertices converging to a polygon P with $n-1$ vertices. In this case, we add a vertex to some side of P to obtain an n-gon P'. The 'new' polygon P' then has n vertices and the 'new' vertex has angle π. Definition 4.12 can now be used to test the convergence of P_k to P'.

It is important to note that our notion of convergence precludes the possibility that two distinct vertices of P_k converge to a single vertex of P'. However, it would be interesting to analyze the convergence of eigenfunctions in this case.

Proposition 4.14 (Stability of the total index)**.** *Suppose that P_n is a sequence of polygons that converges to P, and suppose that $u_n : P_n \to \mathbb{R}$ is a sequence of Neumann eigenfunctions that converge to a Neumann eigenfunction $u : P \to \mathbb{R}$. Let $p \in P$ and suppose that $D \subset \mathbb{C}$ is an open disk neighborhood of p such that ∂D contains no zeros of ∇u. Let A (resp. A_n) denote the set of critical points of u (resp. u_n) that lie in D. If A and A_n are finite, then for each n sufficiently large*

$$\sum_{q\in A} \mathrm{ind}(u,q) \ = \ \sum_{q \in A_n} \mathrm{ind}(u_n, q).$$

Proof. The gradient ∇u_n converges to ∇u, and so the sets A_n converges to A.

First we suppose that p lies in the interior of P. Let γ be a counterclockwise parameterization of ∂D. By Proposition 20.2 in [Taylor], we have that $\sum_{q\in A} \mathrm{ind}(u,q) = \deg(\nabla u/|\nabla u|\circ\gamma)$ and $\sum_{q\in A_n} \mathrm{ind}(u_n,q) = \deg(\nabla u_n/|\nabla u_n|\circ\gamma)$. But the vector field $\nabla u_n/|\nabla u_n|\circ\gamma$ converges to $\deg(\nabla u/|\nabla u|\circ\gamma)$, and hence the degrees converge. Since the degree is an integer invariant, the degrees coincide for all sufficiently large n.

If p lies on the boundary of P, the we apply Lemma 4.10. We have $\sum_{q\in A} \mathrm{ind}(u,q) = \deg(\nabla w/|\nabla w|\circ\gamma)$ and $\sum_{q\in A_n} \mathrm{ind}(u_n,q) = \deg(\nabla w_n/|\nabla w_n|\circ\gamma)$ where w and w_n are constructed as in (20). Since w_n converges to w, the degree of $\nabla w_n/|\nabla w_n|\circ\gamma$ converges to the degree of $\nabla w/|\nabla w|\circ\gamma$. The claim the follows from Lemma 4.10. ☐

Lemma 4.15. *Let P_n be a sequence of polygons that converges to a polygon P and let $u_n : P_n \to \mathbb{R}$ be a sequence of Neumann eigenfunctions that converge to a Neumann eigenfunction $u : P \to \mathbb{R}$. If u has finitely many nonzero index critical points, then there exists N such that if $n > N$, then the number of nonzero index critical points of u_n is greater than or equal to the number of critical points of u.*

Proof. Let p be a nonzero index critical point of P. In particular, p is isolated, and so there exists a disk neighborhood D_p of p that contains no critical points of u other than p. Since $\mathrm{ind}(u,p) \neq 0$, Proposition 4.14 implies that, for sufficiently large n, at least one nonzero index critical points of u_n lies in D_p. Since the various disks D_p are disjoint, the claim follows. □

Lemma 4.16. *Let u be a nonconstant Neumann eigenfunction on a polygon P. If the set of critical points of u is discrete, then each local extremum p of the restriction $u|_{\partial P}$ is a critical point of u.*

Proof. Suppose that p lies in the interior of a side e of p. Since p is a local extremum of $u|_{\partial P}$, we have $L_e u(p) = 0$. Thus, since u satisfies Neumann conditions at p, we have $\nabla u = 0$.

□

5. Index zero critical points on a side of a polygon.

In this section u is a Neumann eigenfunction on a polygon P, and p is an isolated critical point of u that lies in a side e of P. We show in Lemma 5.1 that if $\mathrm{ind}(u,p) = 0$, then the level set $\{z : u(z) = u(p)\}$ is a 'cusp' that is tangent to e (see Lemma 5.1). We then use this to show that if the nodal set of Xu, where X is either a rotational or constant vector field, has a degree 1 vertex, then the vertex is a critical point with nonzero index (Proposition 5.2).

By applying a rigid motion to P we may assume that $p = 0$ and that the side that contains p lies in the real axis.

Lemma 5.1. *Suppose that p is an index zero critical point of a Neumann eigenfunction u that belongs to the side e. Then there exist real-analytic functions $c : \mathbb{C} \to \mathbb{R}$ and $\rho : \mathbb{R} \to \mathbb{R}$ and an odd integer $k \geq 3$ so that $c(0) \neq 0$, $\rho(0) \neq 0$, and*

$$u(z) \;=\; u(0) \;+\; c(z) \cdot \left(y^2 \;- x^k \cdot \rho(x)\right). \tag{21}$$

Proof. Because the index of the critical point p of u equals zero, the Hessian of u at p has exactly one nonzero eigenvalue. Indeed, the Morse lemma implies that nondegenerate critical points have Poincaré-Hopf index equal to 1 or -1. Hence p is degenerate and zero is an eigenvalue of the Hessian of u at p. If the associated eigenspace were two-dimensional, then $\Delta u(p) = 0$, and so p would be a nodal critical point. But the result of [Chn76] shows that a nodal critical point cannot have index zero. Therefore, zero is not the only eigenvalue of the Hessian.

The eigenspace E that corresponds to the nonzero eigenvalue is invariant under the reflection $z \mapsto \overline{z}$. Thus E is either the real or the imaginary axis. We claim that E is not the real axis. Indeed, suppose to the contrary that E is the real axis. Then $\partial_x u(0) = 0$ but $\partial_x^2 u(0) \neq 0$. The Weierstrass preparation theorem applies to provide unique real-analytic functions a, b_1, and b_2 defined near 0, so that $a(0) \neq 0$, $b_1(0) = 0 = b_2(0)$, and

$$u(z) \;-\; u(0) \;=\; a(z) \cdot \left(x^2 \;+\; b_1(y) \cdot x \;+\; b_2(y)\right)$$

for z near $p = 0$. Since the factorization is unique and $u(\overline{z}) = u(z)$, we have $b_j(y) = b_j(-y)$ for $j = 1, 2$. In particular, the discriminant $D(y) := b_1(y)^2 - 4 \cdot b_2(y)$ is an even function. If D were to vanish on a neighborhood of 0, then we would have $u(z) - u(0) = a(z) \cdot (x + b_1(y)/2)^2$ and hence

$$\nabla u(z) \;=\; \left(x + \frac{b_1(y)}{2}\right)^2 \cdot \nabla a(z) \;+\; 2a(z) \cdot \left(x + \frac{b_1(y)}{2}\right) \cdot \nabla \left(x + \frac{b_1(y)}{2}\right).$$

Thus, ∇u would vanish along the level set of u that contains $p = 0$, but by assumption $p = 0$ is an isolated zero of ∇u. Since D is even and $\mathrm{ind}(u, 0) \neq 1$, it follows that $D(y) > 0$ for $y \neq 0$ sufficiently small, and hence there exists a neighborhood U of $p = 0$ so that the intersection of $u^{-1}(u(p)) - \{p\}$ and U consists of four arcs. This contradicts the assumption that p is a zero index critical point of u.

Therefore E coincides with the imaginary axis. By use of the Weierstrass preparation theorem we find that

$$u(z) \;-\; u(0) \;=\; a(z) \cdot \left(y^2 \;+\; b_1(x) \cdot y \;+\; b_2(x)\right) \tag{22}$$

for unique real-analytic functions a, b_1, and b_2 defined near 0 where $a(0) \neq 0$ and $b_1(0) = 0 = b_2(0)$. Since the factorization is unique and $u(\overline{z}) = u(z)$, we find that $b_1(x) = 0$. Because p is an isolated critical point, there exists $\epsilon > 0$ so that $b_2(x) \neq 0$ if $0 < |x| < \epsilon$. We claim that, moreover, $b_2(x) \cdot b_2(-x) < 0$ if $0 < |x| < \epsilon$. Indeed, otherwise $b_2(x) \cdot b_2(-x) > 0$, and thus from (22) we find that there exists a neighborhood U of $p = 0$ so that the intersection of $u^{-1}(u(p)) - \{p\}$ and U consists of four arcs. This contradicts the assumption that p is a zero index critical point of u.

Since $b_2(x) \cdot b_2(-x) < 0$ the first nonzero term in the Taylor series of b_2 about zero has odd degree k, and since $\partial_x u(0) = 0$ we also have $k \geq 3$. The claim follows. □

Proposition 5.2. *Let X be either a constant vector field or a rotational vector field. If p is a degree 1 vertex of $\mathcal{Z}(Xu)$ that is not a vertex of P, then p is a critical point with nonzero index.*

Proof. Since p is not a vertex, p lies in the interior of a side e. Since p is a degree 1 vertex of $\mathcal{Z}(Xu)$, the vector $X(p)$ is independent of the normal vector to ∂P at p, and in particular p is a critical point of u. It remains to show that p has nonzero index.

As above, we may suppose without loss of generality that e lies in the real-axis and that $p = 0$. We will consider the case in which X is a constant vector field of the form $X = \cos(\psi)\partial_x + \sin(\psi)\partial_y$ where $\psi \neq \pi/2 \mod \pi$.

Suppose to the contrary that the index of p were to equal zero. Then by Lemma 5.1, near p, the function u would satisfy (21) where $c(0) \neq 0 \neq \rho(0)$ and $k \geq 3$ is odd. Direct computation shows that $\partial_y Xu(0) = 2\cos(\psi) \cdot c(0)$ and hence $\partial_y Xu(0) \neq 0$. Thus, by the implicit function theorem, there exists a function $f : (-\epsilon, \epsilon) \to \mathbb{R}$ so that

$$Xu\,(x + i \cdot f(x)) \;=\; 0. \tag{23}$$

From (21), we find that for each real x

$$(Xu)(x) \;=\; -\cos(\psi) \cdot c(0) \cdot k \cdot x^{k-1} \;+\; O(|x|^k). \tag{24}$$

By repeatedly differentiating (23) with respect to x and using (24) we find that $\partial_x^j f(0) = 0$ for each $j < k-1$ and $\partial_x^{k-1} f(0) \neq 0$. Since $k-1$ is even and greater than 0, the function f is of one sign in a deleted neighborhood of 0. Thus, there exists a neighborhood U of $p = 0$ such that $(U \cap \mathcal{Z}(Xu)) \setminus \{p\}$ intersects P in either no arcs or two arcs. Hence p is not a degree 1 vertex, a contradiction. □

The following Lemma follows from the discussion in §7 of [JdgMnd20]. We provide a statement and proof for the convenience of the reader.

Lemma 5.3. *Let u be a second Neumann eigenfunction on a polygon P, and let p be an index zero critical point that lies in the side e. If L_e is a constant vector field that is parallel to e, then $\mathcal{Z}(L_e u)$ intersects the interior of P and has at least two degree 1 vertices in $\partial P \setminus e$.*

Proof. By Lemma 5.1, u has the expression

$$u(x, y) \;= u_{00} + u_{02} \cdot y^2 + u_{30} \cdot (x^3 - 3xy^2) \;+\; o(|z|^3)$$

in a neighborhood of p, where $u_{02} \neq 0$. If $u_{30} = 0$ then Proposition 7.4 of [JdgMnd20] provides the claim. If $u_{30} \neq 0$ then the first paragraph of Proposition 7.6 of [JdgMnd20] provides the claim. □

6. Critical points of second Neumann eigenfunctions on simply connected polygons

In this section, we restrict attention to a polygon P that is simply connected and to an eigenfunction u that is associated to the second Neumann eigenvalue.

Proposition 6.1. *The nodal set $\mathcal{Z}(u)$ is a simple arc whose intersection with ∂P consists of its two endpoints. Moreover, the endpoints of this arc lie in distinct sides of P, and $\mathcal{Z}(u)$ does not contain any critical point of u.*

Proof. The first statement is a well-known consequence of Courant's nodal theorem and Polya's inequality.[8] The second statement follows from Lemma 3.3 [JdgMnd20] and Theorem 2.5 in [Chn76]. □

Proposition 6.2. *Let p be a critical point p of a second Neumann eigenfunction u that is not a vertex. Then the index $\mathrm{ind}(u, p)$ equals either $1, 0,$ or -1.*

Proof. Let $\widetilde{u}$ be the lift of u to the double DP, and let $\widetilde{p} \in DP$ correspond to p. If $\mathrm{ind}(u, p) < -1$, then more than four arcs in $\widetilde{u}^{-1}(\widetilde{u}(\widetilde{p}))$ emanate from $\widetilde{p}$. It follows that, in the natural coordinates at $\widetilde{p}$, we have $\widetilde{u}(z) - \widetilde{u}(\widetilde{p}) = o(|z - \widetilde{p}|^2)$. In particular, the degree two homogeneous polynomial h_2 consisting of second order terms in the Taylor expansion of $\widetilde{u}$ at p vanishes indentically. But $(\Delta h_2)(\widetilde{p}) = \mu \cdot \widetilde{u}(\widetilde{p})$ and so $u(\widetilde{p}) = 0$. This contradicts Proposition 6.1. □

Next we consider the possible indices of a critical point of u that lies at a vertex of P. Let c_k be the coefficient in the Bessel expansion (3) at a point $p \in \partial M$. The following should be compared with Corollary 5.3 in [JdgMnd20].

[8] See, for example, Theorem 5.2 [JdgMnd20].

Proposition 6.3. *Let $p \in \partial P$. Either $c_0 \neq 0$ or $c_1 \neq 0$.*

Proof. If $c_0 = 0 = c_1$, then by inspecting (3) one finds that at least two nodal arcs emanate from p. This contradicts Proposition 6.1. □

Corollary 6.4. *Let v be a vertex whose angle β is not a multiple of $\pi/2$.*

(1) If $c_0 = 0$, then $\operatorname{ind}(u, v) = 0$.
(2) If $\beta < \pi/2$, then $c_0 \neq 0$ if and only if v is a local extremum.
(3) If $\beta < \pi$ and $c_1 = 0$, then v is a local extremum.
(4) If $\pi/2 < \beta < \pi$, then $c_1 = 0$ if and only if v is a local extremum.
(5) If $\beta < \pi$, then $\operatorname{ind}(u, v) = 0$ or $\operatorname{ind}(u, v) = 1$.

Proof. Let k be the smallest positive integer such that $c_k \neq 0$. If $c_0 = 0$, then by Proposition 6.3, we have $c_1 \neq 0$, and hence by Lemma 4.7 we have $\operatorname{ind}(u, v) = 0$.

If $\beta < \pi/2$ and $c_0 \neq 0$, then we are in case (ii) of Lemma 4.7, and hence $\operatorname{ind}(u, v) = 1$. By Proposition 4.6 we have $\operatorname{ind}(u, v) = 1$ if and only if v is a local extremum.

Suppose that $\pi/2 < \beta < \pi$. If $c_1 = 0$, then Proposition 6.3 implies that $c_0 \neq 0$, and so part (ii) of Lemma 4.7 implies that $\operatorname{ind}(u, v) = 1$. If $c_1 \neq 0$, then part (i) of Lemma 4.7 implies that $\operatorname{ind}(u, v) = 0$. □

Corollary 6.5. *Suppose that v is an acute vertex of P contained in the side e. If v is not a local extremum, then $\mathcal{Z}(L_e u)$ has an arc that ends at v.*

Proof. This follows from Corollary 2.3 and Corollary 6.4. □

7. No hot spots on certain polygons with two acute vertices

Bañuelos and Burdzy [Bnl-Brd99] used probabilistic methods to show that the second Neumann eigenfunction u of an obtuse triangle has no interior critical points. In [JdgMnd20] [JdgMnd22a], we used a variational approach to show that the two acute vertices are the only critical points of u and hence they are the global extrema of u. In this section, we extend this latter result to a large class of n-gons that have two acute vertices. At this end of the section we identify this class of polygons as those that satisfy the Lip-1 condition of [AtrBrd04] and which have no orthogonal sides.

Lemma 7.1. *Let u be a second Neumann eigenfunction on a simply connected polygon P with at least one acute vertex. If u has an interior critical point, then either u has at least four nonzero index critical points or there is a side e such that $\mathcal{Z}(L_e u)$ has an arc that ends at a vertex of P.*

Proof. Suppose that for each side e, the nodal set $\mathcal{Z}(L_e u)$ does not have an arc that ends at a vertex. Thus, if v is a vertex and e is a side containing v, then Corollary 2.3 implies that the leading Bessel coefficient of u at v is nonzero. In particular, each acute vertex has index $+1$ by Corollary 6.4, and each obtuse vertex is not a critical point by Proposition 4.8.

Since u has a critical point in the interior of P, for any side e, the nodal set $\mathcal{Z}(L_e u)$ has at least two degree 1 vertices in ∂P. Since $\mathcal{Z}(L_e u)$ does not have an arc that ends at a vertex of P, each of the degree one vertices of $\mathcal{Z}(L_e u)$ is a non-vertex point on ∂P. By Proposition 5.2, each of these degree 1 vertices is a nonzero index critical point of u.

Thus, since P has at least one acute vertex, u has at least three nonzero index critical points in ∂P. Since the obtuse vertices are not critical points, Proposition 6.2 implies that each critical point of u that lies in ∂P has index equal to 1, 0 or -1. Thus, it follows from Proposition 4.3 that the number of nonzero index critical points of u that lies on ∂P is even. In particular, u has at least four nonzero index critical points on ∂P. □

In the following we consider paths P_t of polygons with n vertices where the topology on the space of n-gons is given by Definition 4.12. Recall from Remark 4.13 that, by adding a vertex to the side of a polygon, we can consider a polygon with $n - 1$ vertices as a polygon with n vertices. For example, a triangle T may be regarded as a quadrilateral if we declare that some point p on a side of T is a vertex that has angle π.

Let u_t be a path of second Neumann eigenfunctions associated to the path P_t. For each t, let $V(t)$ denote the number of vertices v of P_t with angle not equal to π such that there exists a side e of P_t so that an arc in $\mathcal{Z}(L_e u_t)$ ends at v. Let $S(t)$ denote the number of nonzero index critical points of u_t.

Lemma 7.2. *Suppose that P_t is a path of n-gons such that no two sides of P_t are orthogonal and P_t has exactly two acute vertices for each $t \in [0,1]$. Let u_t be an associated path of eigenfunctions. If $S(0) \geq 3$ or $V(0) \geq 1$, then either $S(1) \geq 3$ or $V(1) \geq 1$.*

Proof. It suffices to show that the set, A, of $t \in [0,1)$ such that either $S(t) \geq 3$ or $V(t) \geq 1$ is both open and closed in $[0,1)$.

(A is open) If $S(t) \geq 3$, then Lemma 4.15 implies that there exists $\epsilon > 0$ such that if $|s-t| < \epsilon$, then $S(s) \geq 3$. Hence, to prove openness, it suffices to assume that $V(t) \geq 1$, and show that there exists $\epsilon > 0$ so that if $|s-t| < \epsilon$ then either $V(s) \geq 1$ or $S(s) \geq 3$.

If $V(t) \geq 1$, then there exists a vertex v of P_t, a side e of P_t, and an arc in $\mathcal{Z}(L_e u)$ that ends at v.

If the leading coefficient at v is nonzero then for s near t the leading coefficient at the corresponding vertex is also nonzero. Since no two sides of P_t are orthogonal and since the corresponding edge and sector vary continuously in s, we find from Lemma 2.1 that $V(s) \geq 1$ for each s near t. Therefore, we may assume that there exists a vertex v such that the leading coefficient of u_t at v equals zero.

We may assume without loss of generality that v is an acute vertex. Indeed, if v were obtuse with $c_1(t) = 0$, then Lemma 4.7 would imply that v is a critical point with nonzero index. If $c_0(t)$ were not to vanish at each of the two acute vertices, then Lemma 4.7 would imply that each of these vertices have index equal to one.

Hence, $S(t) \geq 3$, and so $S(s) \geq 3$ for s near t by Lemma 4.15. Thus, we may assume that v is acute.

Suppose that $c_0(t) = 0$ at an acute vertex v. Corollary 6.4 implies that $\mathrm{ind}(u, v) = 0$. By Proposition 4.16, the eigenfunction u_t has at least two nonzero index critical points. Thus, it follows from Lemma 4.15 that there exists $\epsilon > 0$ such that if $|s-t| < \epsilon$, then there exist two nonzero index critical points of u_s that are distinct from v. Suppose that $0 < |s-t| < \epsilon$. If $c_0(s) \neq 0$, then, since v is acute, Corollary 6.4 implies that $\mathrm{ind}(u_s, v) \neq 0$, and hence $S(s) \geq 3$. On the other hand, if $c_0(s) = 0$, then Corollary 6.5 implies that $\mathcal{Z}(L_e u_s)$ has an arc that ends at v where e is a side adjacent to v. In sum, if $|s-t| < \epsilon$, then either $S(s) \geq 3$ or $V(s) \geq 1$.

(A is closed) By assumption, for each $t \in [0,1)$, no two sides of P_t are orthogonal, and so the set of t such that $V(t) \geq 1$ is closed by Lemma 2.1. Suppose that $S(t_n) \geq 3$ with $t_n \to t$. To prove that A is closed it suffices to show that either $S(t) \geq 3$ or $V(t) \geq 1$.

If the eigenfunction u_t has an interior critical point, then Lemma 7.1 implies that $S(t) \geq 4$ or $V(t) \geq 1$. Thus, we may assume that u_t has no interior critical points. By Proposition 4.16, the two index 1 critical points, p^+ and p^- lie in ∂P. Suppose that there exists a third critical point p. If the index of p is nonzero, then $S(t) \geq 3$. Thus, in the following we assume that $\mathrm{ind}(u, p) = 0$.

If some acute vertex v has is not a local extremum, then by Corollary 6.5 an arc of $\mathcal{Z}(L_e v)$ ends at v, and so $V(t) \geq 1$. Thus, we may assume that each acute vertex is a local extremum. If there are three local extrema, then $S(t) \geq 3$. Hence we may assume that the acute vertices correspond to the the index 1 critical points p^+ and p^-.

Because $S(t_n) \geq 3$, for each n there exists a critical point p_n on a side that is distinct from p^+ and p^-. Suppose that p_n converges to a vertex v of P_t whose angle does not equal π. Then, by Lemma 3.3 the leading coefficient—$c_0(t)$ if v is acute and $c_1(t)$ if v is obtuse—equals zero. If v is obtuse then Corollary 4.8 implies that v is a nonzero index critical point and so $S(t) \geq 3$. If v is acute, then by Corollary 6.5 we have $V(t) \geq 1$.

Therefore, we may assume that p_n converges to a critical point p of u_t that lies in the interior of a side e. Since $p \neq p^{\pm}$, the critical point has index equal to zero. Thus, by Lemma 5.3, the graph $\mathcal{Z}(L_e u_t)$ intersects the interior of P_t and has at least two degree 1 vertices. If one of these degree 1 vertices equals a vertex of P_t then $V(t) \geq 1$. If a degree one vertex lies in the interior of a side then it is a nonzero index critical point by Proposition 5.2, and hence $S(t) \geq 3$ since the acute vertices $p^{\pm}$ are also nonzero index critical points. □

Theorem 7.3. *Suppose that P_t is a path of n-gons such that no two sides of P_t are orthogonal. If P_1 is an obtuse triangle, then each second Neumann eigenfunction of P_0 has exactly two critical points, a global maximum at one acute vertex and a global minimum at the other acute vertex. Moreover, the second Neumann eigenspace of P_0 is one-dimensional.*

Proof. By the method of Lemma 12.2 of [JdgMnd20], one may modify the path P_t so that there exists a continuous family of second Neumann eigenfunctions u_t connecting any u_0 to any u_1. If u_1 is a second Neumann eigenfunction for an obtuse triangle P_1, then by [JdgMnd20] [JdgMnd22a], the acute vertices are the only critical points of u_1, and in particular each is a global extremum and $S(1) = 2$. Thus Proposition 4.6 and Corollary 6.4 imply that the coefficient c_0 of u_1 at each acute vertex is nonzero. Given an acute vertex v, the angle between the opposite side and one of the sides adjacent to v is greater than $\pi/2$. Hence it follows from Lemma 2.1 that for each side e of P_1 there does not exist an arc in $\mathcal{Z}(L_e u)$ that ends at an acute vertex. The obtuse vertex is not a local extremum and hence c_1 of u_1 at this vertex is nonzero. Thus, it follows from Lemma 2.1 that for each side e of P_1, no arc of $\mathcal{Z}(L_e u_1)$ ends at the obtuse vertex. In sum, $S(1) = 2$ and $V(1) = 0$.

Thus, Lemma 7.2 implies that $S(0) = 2$ and $V(0) = 0$. In particular, u_0 has exactly two nonzero index critical points and these are necessarily the global extrema of u_0. Each global extremum must be an acute vertex. Indeed if an acute vertex v of P were not a local extremum, then by Corollary 6.5 we would have that $\mathcal{Z}(L_e v)$ has an arc that ends at v where e is a side adjacent to v, contradicting $V(0) = 0$.

Suppose that there exists a critical point p of u_0 that were distinct from the acute vertices. Then p has index zero and lies in a side of P_0. Thus, p lies in the interior of a side e of P, and hence by Lemma 5.3, the graph $\mathcal{Z}(L_e u_t)$ intersects the interior of P_t and has at least two degree 1 vertices. If a degree 1 vertex p lies in the interior of a side, then $\mathrm{ind}(u, p) \neq 0$ by Proposition 5.2, a contradiction. Therefore, the acute vertices are the only critical points of u_0.

Finally, we show that the second Neumann eigenspace of P_0 is one-dimensional. Let u_+ and u_- be second Neumann eigenfunctions of P_0 and let v be an acute vertex. Then there exist $a_+, a_- \in \mathbb{R}$ so that $a_+ \cdot u_+(v) + a_- \cdot u_-(v) = 0$. We claim that $u^* := a_+ \cdot u_+ + a_- \cdot u_- \equiv 0$. Indeed, if not then u^* would be a second Neumann eigenfunction and in particular would be orthogonal to the constant functions. Thus both the the maximum value and the minimum value of u would be nonzero. But by Theorem 7.3, the acute vertex v is a global extremum of u^* and hence we have a contradiction. $\square$

We now show that the set of polygons that satisfy the hypotheses of Theorem 7.3 is the interior of the set of polygons that satisfy the Lip-1 condition of [AtrBrd04]. First we recall, the notion of Lip-K domain. Let $f_+ : [-b, b] \to \mathbb{R}$ and $f_- : [-b, b] \to \mathbb{R}$ be a pair of Lipschitz functions such that

- $f_+(\pm b) = f_-(\pm b)$,
- $f_-(x) < f_+(x)$ for $x \in (-b, b)$, and
- the Lipschitz constant of $f_\pm$ is at most K.

The domain $\{(x, y) : f_-(x) < y < f_+(x)\}$ is called a *Lip-K domain.*

Recall that if Ω is a domain with Lipschitz boundary $\partial\Omega$ then the outward unit normal vector $\nu(p)$ is defined for almost every $p \in \partial\Omega$.

Proposition 7.4. *A simply connected Lipschitz domain Ω is isometric to a Lip-1 domain if and only if there exists a partition of $\partial\Omega$ into two connected sets Γ^+ and Γ^- so that if $p, p' \in \Gamma^\pm$ then $\nu(p) \cdot \nu(p') \geq 0$ and if $p \in \Gamma^+$ and $q \in \Gamma^-$ then $\nu(p) \cdot \nu(q) \leq 0$.*

Proof. ($\Rightarrow$) After applying an isometry, we may suppose that Ω is bounded by the graphs of the Lip-1 functions f_+ and f_- as above. Let Γ^+ be the graph of f^+ and let Γ^- be the graph of f^-. Suppose that $\nu(p) = (x, y)$. Since f^+ is Lip-1 we have that $p \in \Gamma^+$ implies that $y > |x|$, and since f^- is Lip-1 we have that $p \in \Gamma^-$ implies that $y < -|x|$. It follows that if $p, p' \in \Gamma^\pm$ then $\nu(p) \cdot \nu(p') \geq 0$ and if $p \in \Gamma^+$ and $q \in \Gamma^-$ then $\nu(p) \cdot \nu(q) \leq 0$.

($\Leftarrow$) Let $p_n^+ \in \Gamma^+$ and $p_n^- \in \Gamma^-$ be sequences such that $\lim_{n\to\infty} \nu(p_n^+) \cdot \nu(p_n^-)$ equals the supremum of $\{\nu(p) \cdot \nu(q) : p \in \Gamma^+, q \in \Gamma^-\}$. Let w be a limit point of the sequence $(\nu(p_n^+) - \nu(p_n^-)) / |\nu(p_n^+) - \nu(p_n^-)|$. A computation shows that for each $p \in \Gamma^+$ we have $\nu(p) \cdot w \geq 1/\sqrt{2}$ and for each $p \in \Gamma^-$ we have $\nu(p) \cdot w \leq -1/\sqrt{2}$. Choose coordinates in the plane so that the vector w is the vector $(0, 1)$. Then for each $p \in \Gamma$ we have $\nu(p) = (x, y)$ where $y \geq |x|$. From this it follows that Γ^+ is the graph of a Lip-1 function $f_+ : [a_+, b_+] \to \mathbb{R}$. Similarly, Γ^- is the graph of a Lip-1 function $f_- : [a_-, b_-] \to \mathbb{R}$. Because Γ^+ and Γ^- form a partition of $\partial\Omega$ we have $f_+(a_+) = f_-(a_-)$ and $f_+(b_+) = f_-(b_-)$. Because $\nu(p)$ is the outward normal vector for a domain we have $f_+ > f_-$. $\square$

Corollary 7.5. *A triangle T is a Lip-1 domain if and only if T is not an acute triangle.*

Proof. Let e_1, e_2, e_3 be the sides of the triangle and let ν_1, ν_2 and ν_3 be the associated outward normal vectors. The angle between e_i and e_j is acute if and only if $\nu_i \cdot \nu_j < 0$. The claim follows from Proposition 7.4. $\square$

Proposition 7.6. *Suppose that P_t is a path of polygons such that no two sides of P_t are orthogonal and P_0 is isometric to a Lip-1 domain. Then each P_t is also isometric to a Lip-1 domain.*

Note that we are allowing for the possibility that some vertices have angle π for some t.

Proof. Since P_0 is a Lip-1 domain, there exists a partition $\{\Gamma^+, \Gamma^-\}$ of ∂P_0 that satisfies the criteria of Proposition 7.4. In particular, Γ_+ is the union of sides with outward unit normal vectors $\nu_1^+(0), \ldots, \nu_j^+(0)$, the set Γ_- is the union of sides with outward unit normal vectors $\nu_1^-(0), \ldots, \nu_k^-(0)$, and these normal vectors satisfy $\nu_i^\pm(0) \cdot \nu_j^\pm(0) \geq 0$ and $\nu_i^+(0) \cdot \nu_j^-(0) \leq 0$. Since no two sides of P_0 are orthogonal, each inequality is strict. The quantities $\nu_i^\pm(t) \cdot \nu_j^\pm(t)$ and $\nu_i^+(t) \cdot \nu_j^-(t)$ depend continuously in t and cannot vanish since no two sides of P_t are orthogonal. Thus the inequalities persist for all t, and thus each P_t is a Lip-1 domain by Proposition 7.4. □

Proposition 7.7. *If P is a Lip-1 polygonal domain with no two sides orthogonal, then there exists a path P_t of polygons with no two sides orthogonal such that $P_1 = P$ and P_0 is an obtuse triangle.*

Proof. We will argue via induction on the number, n, of sides of P. If $n = 3$, then the claim follows from 7.5. Suppose that the claim is true if a Lip-1 polygon has n sides no two of which are orthogonal. Let P be a Lip-1 polygon with $n+1$ sides such that no two sides are othogonal. Proposition 7.4 implies that the sides of P can be partitioned into sides $e_1^+, \ldots, e_j^+$ and $e_1^-, \ldots, e_k^-$, so that the associated outward unit normal vectors $\nu_1^+, \ldots, \nu_j^+$ and $\nu_1^-, \ldots, \nu_k^-$ satisfy the inequalities $\nu_i^\pm \cdot \nu_j^\pm > 0$ and $\nu_i^+ \cdot \nu_j^- < 0$. Because P has nonempty interior, by relabeling if necessary, we may assume that $\nu_1^+ \neq \nu_2^+$ and the sides e_1^+ and e_2^+ are adjacent. Let v be the vertex shared by e_1^+ and e_2^+, and let v' be the midpoint of the segment that joins the other two vertices of e_1^+ and e_2^+. Define P_t to be the polygon obtained from P by replacing v with $v_t = (1-t) \cdot v + t \cdot v'$. A straightforward computation show that both $n_1^+(t)$ and $n_2^+(t)$ are convex combinations of n_1^+ and n_2^+, and so it follows that P_t is a Lip 1-polygon with no orthogonal sides. The polygon P_1 may be regarded as a Lip-1 polygon with only n sides no two of which are orthogonal. Thus, by the inductive hypothesis, we may concatenate the path P_t with another path to obtain the desired path to an obtuse triangle. □

8. Instability via blocking

In this section we provide criteria—Proposition 8.1—that guarantee the existence of a quadrilateral with a second Neumann eigenfunction that has an unstable critical point. In §9, we will construct families of quadrilaterals that meet the criteria under the assumption that these quadrilaterals have no interior critical points.

The statement and proof of Proposition 8.1 are somewhat complicated, but the basic idea is simple: Suppose that we have a continuous family of quadrilaterals Q_t with an obtuse vertex w_t and sides e_t^- and e_t^+ adjacent to w_t. Suppose further that for the associated family of eigenfunctions u_t, we know that u_0 (resp. u_1) has only one nonvertex critical point p_0 (resp. p_1), that this critical point lies on the side e_0^- (resp. e_1^+), and that this critical point has index -1. One might naively expect that the index -1 critical point varies continuously in t, and therefore, for some time t, the critical point lies at the obtuse vertex w_t. However, Lemma 3.1 would then imply that $c_1 = 0$ at w_t, and then Corollary 6.4 would imply that w_t is an index $+1$ critical point. Thus, the index of the critical point would abruptly change which is not possible by Proposition 4.14. Roughly speaking, the obtuse vertex 'blocks' the index -1 critical point.

Under additional assumptions, we show that this 'blocking phenomenon' implies the existence of an unstable critical point.

Proposition 8.1. *Let Q_t be a continuous family of quadrilaterals such that for each $t \in [0,1]$ the quadrilateral Q_t has three acute vertices, and the angle of the fourth vertex, w_t, lies in $(\pi/2, \pi)$ for each $t \in (0,1)$. Let e_t be a side of Q_t that is adjacent to w_t so that $t \mapsto e_t$ is continuous. Let $u_t : Q_t \to \mathbb{R}$ be a second Neumann eigenfunction, and suppose that $t \mapsto u_t$ is continuous. Suppose that*

(1) for each t, the eigenfunction u_t has no interior critical points,
(2) for each t, each nonzero index critical point of u_t either is a vertex or belongs to a side adjacent to w_t,
(3) for each t, each acute vertex of Q_t is a local extremum of u_t,
(4) u_0 has exactly one nonvertex critical point and it belongs to the interior of e_0.
(5) u_1 has no critical points on e_1 except for the acute vertex.

Then there exists $t \in (0,1)$ such that u_t has an unstable critical point.

Proof. For each $t \in [0,1]$, let A_t be the set of critical points p of u_t such that either $p = w_t$ or p lies in the interior of a side of Q_t that is adjacent to w_t. We claim that there exists $\delta > 0$ so that for all t no element of A_t is within distance δ of an acute vertex. Indeed, if not, then there would exist $t \in [0,1]$, a sequence $t_n \to t$, and a sequence of critical points p_n of u_n that converges to an acute vertex v. Lemma 3.1 would then imply that $c_0 = 0$ at v, but this would contradict part (3) of Corollary 6.4 and condition (3) above.

By condition (4), the set A_0 has exactly one element p_0, and it follows from Proposition 4.3 that the index of p_0 equals -1. Thus, Proposition 4.14 implies that the sum of the indices of the critical points in A_t equals -1.

Let t^* be the supremum of $t \in [0,1]$ such that A_s contains exactly one nonzero index point, p_s, for each $s \leq t$. It follows from Proposition 4.14 that $s \mapsto p_s$ is continuous on $[0, t^*)$ and the index of each p_s equals -1. Moreover, as $s \nearrow t^*$ the point p_s converges to a point p_{t^*}. Indeed, if p_t were to have more than one limit point as $t \nearrow t^*$, then, since $t \mapsto p_t$ is continuous for $t < t^*$, we would have a nontrivial continuum of critical points. But since Q_{t^*} is not a rectangle, the function u_{t^*} has only finitely many critical points [JdgMnd22b].

Proposition 4.14 implies that the index of p_{t^*} equals -1. It follows that the critical point p_{t^*} lies in the interior of e_{t^*}. Indeed, otherwise, condition (5) would imply that $p_s = w_s$ for some $s \leq t^*$. But this would contradict part (5) of Corollary 6.4.

If A_{t^*} contains a critical point q that is distinct from p_{t^*} then q is an unstable critical point since p_s is the only critical point in A_s for $s < t^*$. For the remainder of the proof we will suppose that p_{t^*} is the only element of A_{t^*}.

By the definition of t^*, there exists a sequence $t_n \searrow t^*$ such that A_{t_n} consists of more than one nonzero index critical point. Since p_{t^*} is the only critical point in A_{t^*} these points converge to p_{t^*}, and in particular for n sufficiently large, the set A_{t_n} lies in the interior of e_{t_n}. By Proposition 6.2, each nonzero index critical point has index $+1$ or -1. Hence since the sum of the indices equals -1, the set A_{t_n} contains at least three critical points.

But this is impossible. Indeed, if three critical points of u_{t_n} were to lie in the interior of e_{t_n}, then $\mathcal{Z}(L_{e_{t_n}} u_{t_n})$ would have three degree 1 vertices that lie in $\partial Q_{t_n} \setminus e_{t_n}$, and in particular some degree 1 vertex would lie in the interior of a side not adjacent to w_{t_n}. But this degree 1 vertex would be a nonzero index critical point by Proposition 5.2, thus contradicting (2).

□

9. Breaking acute triangles along a side

In this section, we will construct families of quadrilaterals that satisfy the hypotheses (2) through (5) of Proposition 8.1. The construction consists of:

1. Producing a nonempty open set $\mathcal{N}$ of acute triangles T such that the set of critical points of each second Neumann eigenfunction u consists only of the three vertices and an index -1 critical point;
2. 'Breaking' the side that contains the index -1 critical point of $T \in \mathcal{N}$ to create quadrilaterals for which each acute vertex is a critical point and for which the only sides that may contain critical points in their interior are the sides adjacent to the new obtuse vertex;
3. Choosing a path w_t of break points so that the resulting path Q_t of quadrilaterals forces 'blocking' to occur.

We now provide the details of this construction. Define $\mathcal{N}$ to be the set of acute triangles T such that if u is any second Neumann eigenfunction on T, then

(1) each vertex of T is a local extremum of u,
(2) u has exactly one nonvertex critical point p.
(3) the critical point p is nondegenerate.

Proposition 4.3 implies that p has index equal to -1. The main theorem of [JdgMnd22a] implies that p lies on a side of T. Note that equilateral triangles do not belong to $\mathcal{N}$, and hence by a result of Siudeja [Siudeja], the second Neumann eigenspace of T is one dimensional for each $T \in \mathcal{N}$.

Lemma 9.1. *The set $\mathcal{N}$ is open in the space of acute triangles.*

Proof. By Corollary 6.4, a vertex v of an acute triangle is a local extremum if and only if $u(v) \neq 0$. Thus, condition (1) is open. A critical point p is nondegenerate if and only if the determinant of the Hessian at p is nonzero, and hence condition (3) is also an open condition.

Thus, if $\mathcal{N}$ were not open, then there would exist $T \in \mathcal{N}$ and a sequence T_n converging to T such that condition (2) is not satisfied for each n. In particular, for each n there would exist a second Neumann eigenfunction u_n on T_n with distinct nonvertex critical points p_n and q_n.

By passing to a subsequence if necessary, we may assume without loss of generality that u_n converges to an eigenfunction u on T. Neither of the sequences p_n nor q_n can converge to a vertex of T because then, by Lemma 3.1, we would have $c_1 = 0$ contradicting (1). Thus, by (2), both sequences converge to the unique nonvertex critical point p of T, and it would follow that p is a degenerate critical point, contradicting (3). □

The set $\mathcal{N}$ is also nonempty.

Lemma 9.2. *Let T be an isosceles triangle with reflection symmetry σ, and let u be a second Neumann eigenfunction of T. If the angle of the apex vertex v fixed by σ is less than $\pi/3$, then*

(1) each vertex is a local extremum of u,
(2) u has exactly one non-vertex critical point p, the midpoint of the side e opposite to v,
(3) p is nondegenerate with index -1,
(4) $u(z) \neq 0$ for each $z \in e$.

Proof. By Lemma 3.1 in [Miyamoto] the second Neumann eigenvalue of T has multiplicity one, and u is symmetric with respect to σ. It follows that the the midpoint p of the side e preserved by σ is a critical point. Let $T_+, T_- \subset T$ be the two right triangles such that $\sigma(T_+) = T_-$ and $T_+ \cup T_- = T$. Since u is symmetric with respect to σ, the restriction of u to $T_\pm$ is a second Neumann eigenfunction of $T_\pm$.

By Theorem 4.1 in [JdgMnd22a], the restriction of u to the right triangle $T_\pm$ has no nonvertex critical points and each acute vertex of $T_\pm$ is a local extremum. It follows that each vertex of T is a local extremum of u and the midpoint p of e is the only other critical point of u. Thus, Theorem 4.3 implies that p has index -1.

Next we show that u does not vanish on e. By Proposition 6.1, the nodal set $\mathcal{Z}(u)$ does not contain a critical point and hence does not contain the midpoint p. Thus, if there did exist $z \in e$ with $u(z) = 0$, then $z \neq p$ and hence $\sigma(z) \neq z$. Since u is symmetric, we would have $u(\sigma(z)) = 0$ but then $\sigma(z)$ would be a second endpoint of $\mathcal{Z}(u)$ that lies in e, a contradiction. Therefore, u does not vanish on e.

Finally, by examining the Taylor expansion of u about p, we find that ~~the~~ p is non-degenerate. Indeed, without loss of generality, $p = 0$ and e lies in the x-axis. Since $u \circ \sigma = u$, the restriction of u to e is an even function of x. In particular, the Taylor coefficient $a_{30} = 0$. Thus, if p were degenerate, then Theorems 7.3 and 7.4 in [JdgMnd20] would imply that u has an additional non-vertex critical point, a contradiction. □

Next, we will 'break' each $T \in \mathcal{N}$ along the side that contains the index -1 critical point. We first give a precise definition of 'breaking': Let T be a triangle[9] with vertices v_1, v_2, v_3. Let e be a side of T, let w be a point that lies in the interior of e, and let n_w be the outward pointing unit normal vector at w. For each $\epsilon \geq 0$, define $w(\epsilon) = w + \epsilon \cdot n_w$, and define $Q(T, w, \epsilon)$ to be the convex hull of $\{v_1, v_2, v_3, w(\epsilon)\}$. For $\epsilon > 0$, the polygon $Q(T, w, \epsilon)$ is a nondegenerate quadrilateral. We say that $Q(T, w, \epsilon)$ *is the result of breaking T along e at the point w at distance ϵ.*

Lemma 9.3. *Let $T \in \mathcal{N}$ and let e be the side of T that contains the index -1 critical point. Let K be a compact subset of the interior of e. There exists $\delta > 0$ such that if $0 \leq \epsilon < \delta$ and $w \in K$, then*

(a) the second Neumann eigenfunction u of $Q(T, w, \epsilon)$ is unique up to scalar multiplication,
(b) each acute vertex of $Q(T, w, \epsilon)$ is a local extremum of u,
(c) if e' is a side that does not contain the obtuse vertex w, then the interior of e' does not contain a critical point of u.

Proof. The simplicity of the second Neumann eigenvalue is an open condition, and the second eigenvalue of each $T \in \mathcal{N}$ is simple by [Siudeja]. It follows that there exists $\delta' > 0$, so that (a) holds for each $Q(T, w, \epsilon)$ with $\epsilon < \delta'$ and $w \in K$. Corollary 6.4 implies that condition (b) is an open condition. In particular, $c_0 \neq 0$ at each acute vertex.

Thus, if the claim were false, then there would exist a sequence $\epsilon_n \to 0$ and $w_n \in K$ such that $Q_n := Q(T, w_n, \epsilon_n)$ has a second Neumann eigenfunction u_n with a nonvertex critical point p_n on a side e' that does not contain $w_n(\epsilon_n)$. The sequence Q_n converges to T, and thus by passing to a subsequence if necessary, we may assume that u_n converges to an eigenfunction u on T. If the sequence $p_n \in e'$ were to converge to a vertex v of T, then Lemma 3.1 would imply $c_1 = 0$ at v, a contradiction. If the sequence p_n converges to a point p in the interior of e', then p is a critical point of u, contradicting the assumption that the 'unbroken' sides of T contain no critical points. □

Let $\delta_{T,K}$ denote the supremum of all possible δ for which the statement of Proposition 9.3 is true for the given compact set K.

Lemma 9.4. *Let $T \in \mathcal{N}$ and let e be the side of T that contains the index -1 critical point p. Let w_t be a path in the interior of e so that w_0 and w_1 lie in distinct components of $e \setminus \{p\}$. If K is the image of the path w_t, then for each $\epsilon \in (0, \delta_{T,K})$, the path $Q_t := Q\left(T, w_t, \epsilon \cdot \sin(t \cdot \pi)\right)$ has an associated path u_t of second Neummann eigenfunctions that satisfy the conditions (2) through (5) of Proposition 8.1.*

[9]One can easily extend the notion of breaking along a side to general polygons.

Proof. By the defintion of $\delta_{T,K}$, the quadrilateral $Q(T, w_t, \epsilon)$ satisfies (a), (b), and (c) of Lemma 9.3. Condition (a) implies that there exists a path u_t of eigenfunctions of Q_t. Condition (b) implies that u_t satisfies condition (3) in Proposition 8.1, and condition (c) implies that condition (2) is satisfied.

Let e_t be the side so that e_0 is the component of $e \setminus \{p\}$ that contains p. It follows that conditions (4) and (5) of Proposition 8.1 are satisfied. □

Theorem 9.5. *Suppose that each convex quadrilateral has no interior critical points. Let $T \in \mathcal{N}$ and let e be the side of T that contains the index -1 critical point. Then for each $\eta > 0$ there exists $\epsilon \in (0, \eta)$ and w in the interior of e so that each second Neumann eigenfunction u of $Q(T, w, \epsilon)$ has an unstable critical point.*

Proof. Lemma 9.4 provides us with a family of quadrilaterals Q_t and second Neumann eigenfunctions u_t that satisfy conditions (2) through (5) of Proposition 8.1. If each second Neumann eigenfuction on a quadrilateral were to have no interior critical points, then each u_t would also satisfy condition (1). Therefore, Proposition 8.1 would imply that for some t the function u_t has an unstable critical point. □

10. Second Neumann eigenfunctions on convex polygons

Proposition 10.1. *Suppose that P is convex without right angles and suppose that w lies in the interior of P. An arc of $\mathcal{Z}(R_w u)$ ends at a vertex v of P if and only if v is a local extremum of u.*

Proof. If w lies in P, then it lies in the interior of the sector associated to v. By assumption the angle at v lies in either $(0, \pi/2)$ or $(\pi/2, \pi)$. The claim then follows from combining Corollary 2.5, Corollary 4.8, and Corollary 6.4. □

With additional hypotheses, we can expand the scope of Proposition 5.2 to include degree 1 vertices of $\mathcal{Z}(R_w u)$ that are vertices of P.

Corollary 10.2. *Let u be a second Neumann eigenfunction of a convex polygon P with no right angles. If w lies in the interior of P then each degree one vertex of $\mathcal{Z}(R_w u)$ is a nonzero index critical point.*

Proof. Each degree 1 vertex p of $\mathcal{Z}(R_w u)$ lies in ∂P. If p lies in the interior of an edge, then Proposition 5.2 applies. If p is a vertex, then Proposition 10.1 applies. □

Proposition 10.3. *Let u be a second Neumann eigenfunction u on a convex polygon P. If u has a critical point p that lies in the interior of P, then u has at least four nonzero index critical points on the boundary. In particular, u has at least five critical points.*

Proof. Without loss of generality $p = 0$. Since p is a critical point of u we have

$$u(z) \;=\; u(0) \;+\; a \cdot x^2 + b \cdot xy + c \cdot y^2 \;+\; O(|z|^3)$$

for some constants a, b and c. We have $R_p u = -y\partial_x + x\partial_y$ and hence

$$R_p u(z) \;=\; b \cdot (x^2 - y^2) \;+\; 2(c-a) \cdot xy \;-\; b \cdot y^2 \;+ O(|z|^3).$$

In particular, $p = 0$ is a nodal critical point of the Laplace eigenfunction $R_p u$. Thus, by the result of [Chn76], the valence of $\mathcal{Z}(R_p u)$ at p is at least four. By Proposition 6.2 in [JdgMnd20], the nodal set $\mathcal{Z}(R_p u)$ is a tree whose degree 1 vertices lie in the boundary of P. Thus $\mathcal{Z}(R_p u)$ has at least four degree 1 vertices, and each of these is a nonzero index critical point by Corollary 10.2. □

Corollary 10.4. *If u has has only three critical points, then each critical point lies on the boundary. Moreover, one critical point is a global maximum, one critical point is a global minimum, and the third critical point has index zero.*

Proof. By Proposition 10.3, each critical point lies on the boundary. Since u is nonconstant, at least two of these critical points are global extrema. The index of each global extremum is $+1$. Thus, if there are exactly three critical points, then it follows from Proposition 4.3 that two critical points have index 1 and the third has index zero. □

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