EUCLIDEAN TRIANGLES HAVE NO HOT SPOTS

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Abstract. We show that a second Neumann eigenfunction u of a Euclidean triangle has at most one (non-vertex) critical point p, and if p exists, then it is a non-degenerate critical point of Morse index 1. Using this we deduce that
(1) the extremal values of u are only achieved at a vertex of the triangle, and
(2) a generic acute triangle has exactly one (non-vertex) critical point and that each obtuse triangle has no (non-vertex) critical points.
This settles the 'hot spots' conjecture for triangles in the plane.

1. Introduction

Let Ω be a domain in Euclidean space with Lipschitz boundary. The second Neumann eigenvalue, μ_2, is the smallest positive number such that there exists a not identically zero, smooth function $u : \Omega \to \mathbb{R}$ that satisfies

$$\Delta u \ = \ \mu_2 \cdot u \quad \text{and} \quad \left. \frac{\partial u}{\partial n} \right|_{\partial\Omega} \equiv \ 0 \tag{1}$$

where $\frac{\partial}{\partial n}$ denotes the outward pointing unit normal vector field defined at the smooth points of $\partial\Omega$. A function u that satisfies (1) will be called a *second Neumann eigenfunction for* Ω, or simply a μ_2*-eigenfunction*.

One variant of the 'hot spots' conjecture, first proposed by J. Rauch at a conference in 1974,[1] asserts that a second Neumann eigenfunction attains its extrema at the boundary. The main result of this paper implies the hot spots conjecture for triangles in the plane.

Theorem 1.1. *If u is a second Neumann eigenfunction for a Euclidean triangle T, then u has at most one critical point.*[2] *Moreover, if u has a critical point p, then p lies in a side of T and p is a nondegenerate critical point with Morse index equal to* 1.

In Theorem 12.4, we show that if T is a generic acute triangle, then u has exactly one critical point, and that if T is an obtuse triangle, then u has no critical points. Earlier, Bañuelos and Burdzy showed that if T is obtuse, then u has no interior maximum, and, in particular, the maximum and minimum values of u are achieved at the acute vertices [Bnl-Brd99]. We extend the latter statement to all triangles (see Theorem 12.1). Unlike [Bnl-Brd99], our proof of Theorem 1.1 does not rely on probabilistic techniques.

For a brief history and various formulations of the 'hot spots' conjecture, we encourage the reader to consult [Bnl-Brd99]. We provide some highlights. The first positive result towards this conjecture was due to Kawohl [Kwl85] who showed that the conjecture holds for cylinders in any Euclidean space. Burdzy and Werner in [Brd-Wrn99] (and later Burdzy in [Brd05]) showed that the conjecture fails for domains with two (and one) holes. In the paper [Brd05] Burdzy made two separate ('hot spot') conjectures for 'convex' and 'simply connected' domains. We believe that the conjecture is true for all convex domains in the plane.

The conjecture has been settled for certain convex domains with symmetry. In 1999, under certain technical assumptions, Bañuelos and Burdzy [Bnl-Brd99] were able to handle domains with a line of symmetry. A

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[1] See [Rch74] for a discussion of hot spots.

[2] We do not consider a vertex of a triangle to be a critical point.

year later Jerison and Nadirashvili [Jrs-Ndr00] proved the conjecture for domains with two lines of symmetry. In a different direction, building on the work in [Bnl-Brd99], Atar and Burdzy [Atr-Brd04] proved the conjecture for *lip domains* (a domain bounded by the graphs of two Lipschitz functions with Lipschitz constant 1). In 2012, the hot spots conjecture for acute triangles became a 'polymath project' [Polymath]. In 2015 Siudeja [Sdj15] proved the conjecture for acute triangles with at least one angle less than $\pi/6$ by sharpening the ideas developed by Miyamoto in [Mym09, Mym13]. Notably, in the same paper, Siudeja proved that the second Neumann eigenvalue of an acute triangle T is simple unless T is an equilateral triangle. An earlier theorem of Atar and Burdzy [Atr-Brd04] gave that the second Neumann eigenvalue of each obtuse and right triangle is simple.

Our approach to the conjecture differs from most of the previous approaches (but has some features in common with the approach in [Jrs-Ndr00]). For each acute or obtuse triangle, T_0, we consider a family of triangles T_t that joins T_0 to a right isosceles triangle T_1. Using the simplicity of μ_2 (due to [Atr-Brd04], [Mym13] and [Sdj15]) we then consider a family of second Neumann eigenfunctions associated to T_t.[3] Because T_1 is the right isosceles triangle, the function u_1 is explicitly known up to a constant, and a straightforward computation shows that u_1 has no critical points (see equation (21)). Therefore, if u_0 were to have a critical point, then it would have to somehow 'disappear' as t tends to 1. Each nondengenerate critical can not disappear immediately, that is, it is 'stable'. On the other hand, a degenerate critical point can instantaneously disappear, that is, it could be 'unstable'. Thus, as t varies from 0 to 1, either a critical point p_t of u_t converges to a vertex or p_t is or becomes degenerate and then disappears. Understanding the first case, among the last two possibilities, is more or less straightforward, and we do it by studying the expansion of u_t in terms of Bessel functions near each vertex. Understanding the second case is more complicated. One particular reason for this complication is that disappearance of this type probably does occur for perturbations of general domains.

The study of how eigenvalues and eigenfunctions vary under perturbations of the domain is a classical topic (see for example [Kato]). Jerison and Nadirashvilli [Jrs-Ndr00] considered one-parameter families of domains with two axes of symmetry and studied how the nodal lines of the directional derivatives of the associated eigenfunctions varied. In particular, they used the fact that each constant vector field L commutes with the Laplacian, and hence if u is an eigenfunction, then Lu is also an eigenfunction with the same eigenvalue. The eigenfunctions Lu were also used in [Sdj15] and implicitly in [Bnl-Brd99] and [Atr-Brd04]

In the current paper, we consider the vector field R_p, called the *rotational vector field*, that corresponds to the counter-clockwise rotational flow about a point p. To be precise if $p = p_1 + ip_2$, then

$$R_p \ = \ -(y - p_2) \cdot \partial_x \ + \ (x - p_1) \cdot \partial_y.$$

We will call $R_p u$ the *angular derivative of* u *about* p. Each rotational vector field R_p commutes with the Laplacian, and hence the angular derivative $R_p u$ is an eigenfunction. By studying the nodal sets of $R_v u$ where v is a vertex v of the triangle, one finds that if u has an interior critical point, then u also has a critical point p on each side of the triangle (see Corollary 6.2). Moreover, we show that each of these three critical points is stable under perturbation even though p might not be stable (see Proposition 9.4). We also use the nodal sets of both $R_p u$ and $L_e u$, where L_e is parallel to the side e of T, to show that, although a degenerate critical point of u might not be stable under perturbation, there are at least two other critical points that are stable under perturbation (see Proposition 9.5).

Outline of the paper. In §2, we recall Cheng's [Chn76] theorem concerning the structure of the nodal set of an eigenfunction on surfaces. From a result of Lojasiewicz [Ljs59] it follows that the critical set of each eigenfunction is a disjoint union of isolated points and analytic one-dimensional manifolds. In §3, we consider domains obtained from a triangle via reflecting about its sides. By applying Cheng's structure theorem to the extension of an eigenfunction to these extended polygonal domains, we obtain a qualitative result concerning nodal arcs whose endpoints lie in a side of the triangle. In §4, we consider the Bessel expansion of a Neumann

[3] In fact, one can avoid using the simplicity of the second eigenvalue. See §11.

eigenfunction on a sector. Using the radial and angular derivatives of this expansion, we obtain a qualitative description of the critical set of a Neumann eigenfunction on a sector. We use this lower estimate in §5 to prove that the critical set of a second Neumann eigenfunction u on a triangle T is finite. There we also (re)prove the fact that the nodal set of u is a simple arc, and use this fact to obtain information about the first two Bessel coefficients of u at the vertices of T. For example, we deduce that u can vanish at only one vertex of T. In §6, we study the nodal set of both the angular derivatives, $R_v u$, about vertices v and the directional derivatives, $L_e u$, parallel to an edge e. We show that each component of each of these nodal sets is a finite tree, and use this to obtain information about the critical set of u. For example, we show that if u has an interior critical point then it has at least three more critical points, one critical point per side (Corollary 6.2), and if u has a degenerate critical point on a side e, then u has a critical point on a side distinct from e (Theorem 6.5).

In §8, we begin the proof of Theorem 1.1. Given an obtuse or (non-equilateral) acute triangle T_0, we consider a 'straight line path' of triangles T_t that joins T_0 to a right isosceles triangle and an associated path $t \mapsto u_t$ of second Neumann eigenfunctions. In §8, we suppose t_n converges to t and consider the accumulation points of a sequence p_n where each p_n is a critical points of u_{t_n}. Using the Bessel expansion of u_{t_n}, we find that if each p_n lies in the interior of T_{t_n}, then a vertex is not an accumulation point of p_n. We also show that if each p_n lies in a side e and a vertex v is an accumulation point of p_n, then there does not exist a sequence of critical points q_n lying in a distinct side that has v as an accumulation point.

In §9, we address the issue of the stability of critical points. We regard a critical point p of u_t as 'stable' if for each neighborhood U of p the function u_s has a critical point in U for s sufficiently close to t. Non-degenerate critical points are stable, but, in general, degenerate critical points are not. Nonetheless, we use the results of §6 to show that if p is a degenerate critical point of u_t, then u_t has at least two stable critical points. In §10, we use the existence of these two stable critical points to show that, if u_0 has an interior critical point, then u_t also has at least two critical points for each $t < 1$ that is near 1. In contrast, the eigenfunction u_1 for the right isosceles triangle has no critical points, and thus, to prove Theorem 1.1 for acute and obtuse triangles, it suffices to show that the number of critical points can not drop from two to zero in the limit as t tends to one. This is accomplished by using the results of §8 and certain elementary properties of u_1.

To make the exposition of the proof of Theorem 1.1 easier, we use the known simplicity of the second Neumann eigenvalue [Bnl-Brd99] [Atr-Brd04] [Mym13], [Sdj15]. However, we indicate in §11 how to avoid this assumption.

In §12, we use the topology of the nodal sets of the extension of u to the double of the triangle to show that u has a critical point if and only if each vertex is an isolated local extremum of u. In particular, if u is associated to an acute triangle, then u has a critical point if and only if u does not vanish at any of the vertices. In the final part of §12, we consider the parameter space $\mathcal{T}$ of all labeled triangles up to homothety. Using analytic perturbation theory and Hartog's separate analytic theorem we deduce that the Bessel coefficients of a second Neumann eigenfunction (at a labeled vertex) can be thought of (in a suitable sense) as analytic functions on a dense open subset of $\mathcal{T}$. Using this fact, we deduce that a generic acute triangle has exactly one critical point and obtuse triangles have no critical points.

Notation and terminology. For notational convenience, we will regard the Euclidean plane as the complex plane. That is, we will use $z = x + iy$ to represent a point in the plane. In particular, $x = \mathrm{Re}(z)$ and $y = \mathrm{Im}(z)$, and if $z = re^{i\theta}$ then $\theta = \arg(z)$ and $r = |z|$. We will also use $\mathcal{Z}(f)$ to denote the set of z such that $f(z) = 0$, and A° to denote the interior of a set A. For us, a Laplace eigenfunction φ is a smooth real valued solution to the equation $\Delta\varphi = \lambda \cdot \varphi$ where $\Delta = -(\partial_x^2 + \partial_y^2)$ and $\lambda \in \mathbb{R}$. We will sometimes call such a solution φ a λ-eigenfunction.

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//youtu.be/bO5OjFOxCAw. He created these contour plots with his 'fe.py' python script [Clm16]. We also thank David Jerison and Bartlomiej Siudeja for comments on the first version of the paper.

2. The nodal set and the critical set of an eigenfunction

Let $\Omega \subset \mathbb{C}$ be an open set, and let $\varphi : \Omega \to \mathbb{R}$ be an eigenfunction of the Laplacian. In this section, we recall some facts about the nodal set $\mathcal{Z}(\varphi) := \varphi^{-1}(0)$ and the set, $\mathrm{crit}(\varphi)$, of critical points of φ. The intersection $\mathcal{Z}(\varphi) \cap \mathrm{crit}(\varphi)$ is the set of *nodal critical points*.

The following is a special case of the stratification of real-analytic sets due to Lojasiewicz [Ljs59]. An elementary proof can be found in the proof of Proposition 5 in [Otl-Rss09].

Lemma 2.1. *Let Ω be an open subset of $\mathbb{C}$. If $f : \Omega \to \mathbb{R}$ is a real-analytic function, then each $z \in f^{-1}(0)$ has a neighborhood U such that $U \cap f^{-1}(0)$ is either equal to $\{z\}$ or is homeomorphic to a properly embedded finite graph. Moreover, if $\nabla f(z) \neq 0$, then $U \cap f^{-1}(0)$ is a real-analytic arc.*

Because the Laplacian is a constant coefficient elliptic operator, the eigenfunction φ is real-analytic function. Therefore, it follows from Lemma 2.1 that $\mathcal{Z}(\varphi)$ is a locally finite graph whose vertices are the nodal critical points, and the complement of these vertices is a disjoint union of real-analytic loops and arcs. Cheng observed [Chn76] that (in dimension 2) the nodal set has a special structure in a neighborhood of each nodal critical point.

Lemma 2.2 (Theorem 2.5 in [Chn76])**.** *Let φ be an eigenfunction of the Laplacian on an open set $\Omega \subset \mathbb{C}$. If $p \in \Omega$ is a nodal critical point, then there exist a neighborhood U of p, a positive integer $n \geq 2$, a real number θ, and simple C^1 arcs $\{\alpha_1, \dots, \alpha_n\}$, such that*

1. $\bigcap_{i=1}^n \alpha_i = \{p\}$,
2. *$\mathcal{Z}(\varphi) \cap U$ equals $\bigcup_{i=1}^n \alpha_i$, and*
3. *for each i, the arc α_i is tangent at p to the line $\left\{z : \arg(z-p) = \frac{i}{n} \cdot \pi + \theta\right\}$.*

Remark 2.3. Arcs satisfying condition (3) of Lemma 2.2 are called *equiangular*.

Sketch of proof. Without loss of generality $p = 0$. The Taylor series of φ about p may be regarded as a sum $\sum_k h_k$ of homogeneous polynomials h_k of degree k in x and y. Because p is a nodal critical point, h_1 and h_2 vanish identically. Since φ is an eigenfunction and Δ maps homogeneous polynomials of degree k to homogeneous polynomials of degree $k-2$, we have $\Delta h_k = \lambda \cdot h_{k-2}$. In particular, if n is the smallest n such that $h_n \neq 0$, then $\Delta h_n = 0$. Thus, $\varphi = h_n + O(n+1)$ where h_n is a harmonic polynomial of degree at least n and $O(k)$ denotes a sum of terms of degree at least $k+1$. The restriction of the harmonic polynomial h_n to the unit circle centered at $p = 0$ is a Laplace eigenfunction with eigenvalue $(\pi n)^2$, and so since h_n is homogeneous, the nodal set of h_n equals the union of lines $\{z : \arg(z) = \frac{i}{n} \cdot \pi + \theta\}$ for some $\theta \in \mathbb{R}$. One obtains the claim by applying the method of [Kuo69]. See Lemma 2.4 in [Chn76]. □

As a consequence of Lemma 2.2, the nodal set $\mathcal{Z}(\varphi)$ is the union of C^1 loops and proper[4] C^1 arcs. We will call these the *Cheng curves* of φ.

We shall be interested in whether certain Cheng arcs cross a line or not. To make this precise, we note the following.

Lemma 2.4. *Let α be a Cheng curve in $\mathcal{Z}(\varphi)$ and let p be an intersection point of α and a line ℓ. There exists an open neighborhood U of p and a parameterization $c : (-\epsilon, \epsilon) \to U$ of $\alpha \cap U$ such that $c(0) = p$ and either*

1. *the sets $c((-\epsilon, 0))$ and $c((0, \epsilon))$ lie in different components of $\Omega \setminus \ell$,*
2. *the sets $c((-\epsilon, 0))$ and $c((0, \epsilon))$ lie in the same component of $\Omega \setminus \ell$ or*
3. *the curve α lies in the component of $\ell \cap \Omega$ that contains p.*

[4] By 'proper', we mean that the arc can be parameterized by a proper map $\alpha : \mathbb{R} \to \Omega$.

In case (1), we say that the curve α *crosses* the line ℓ.

Proof. The restriction of φ to ℓ is a real-analytic function on $\Omega \cap \ell$. We have case (3) if and only if this restriction vanishes identically on the component containing p. If it it does not vanish identically, then there exists a neighborhood U of p such that U contains no zeros of the restriction other than p. Choose a C^1 parameterization c of α so that $c((\epsilon, 0))$ and $c((0, \epsilon))$ do not intersect ℓ. □

The set, $\mathrm{crit}(\varphi)$, of critical points has the following description parallel to that of the nodal set $\mathcal{Z}(\varphi)$.

Proposition 2.5. *Let φ be a Laplace eigenfunction on an open set $\Omega \subset \mathbb{R}^2$. Each connected component of* $\mathrm{crit}(\varphi)$ *is either*

(1) *an isolated point,*
(2) *a proper real-analytic arc, or*
(3) *a real-analytic curve that is homeomorphic to a circle.*

Proof. The function $f = |\nabla \varphi|^2$ is analytic, and hence by Lemma 2.1 each critical point is either isolated or lies in a component of $\mathrm{crit}(\varphi)$ that is a locally finite graph.

Let A be a component of the graph $\mathrm{crit}(\varphi)$. If $\Delta\varphi(z) = 0$ for some $z \in A$, then since A is connected and $\nabla\varphi = 0$ on A, it would follow that $A \subset \mathcal{Z}(\varphi)$. By Lemma 2.2, the set of nodal critical points is discrete, and hence A would consist of an isolated point.

If $\Delta\varphi(z) \neq 0$, either $\partial_x^2 \varphi(z) \neq 0$ or $\partial_y^2 \varphi(z) \neq 0$. Without loss of generality, we may assume that $\partial_x^2\varphi(z) \neq 0$, and hence $\nabla(\partial_x \varphi)(z) \neq 0$. Therefore, the analytic implicit function theorem provides a neighborhood U_z of z such that $\{w : \partial_x \varphi(w) = 0\} \cap U_z$ is a real-analytic arc α. The set $A \cap U_z$ lies in α.

By Lemma 2.1, the set $A \cap U_z$ is either finite, and hence A is an isolated point, or $A \cap U_z$ is a proper finite graph. In the latter case $A \cap U_z = \alpha$. Since $z \in A$ is arbitrary, the component A is a real-analytic 1-manifold (without boundary). If A is compact, then A is homeomorphic to a circle. Otherwise, there exists a possibly infinite open interval $I \subset \mathbb{R}$ and a real-analytic unit speed parameterization $\gamma : I \to A$. Since A is closed in U, the map γ is proper. □

3. Eigenfunctions on triangles, kites, and hexagons

In this section we consider eigenfunctions on the triangle $T \subset \mathbb{C}$ that satisfy Neumann conditions along at least one of the sides of T. Let e be a side of T, and let $\sigma_e : \mathbb{C} \to \mathbb{C}$ denote the reflection across the line containing e. Following [Sdj15], we define the *kite* K_e to be the closed set $T \cup \sigma_e(T)$. If φ is an eigenfunction of the Laplacian that satisfies Neumann conditions along e, then φ extends uniquely to a real-analytic Neumann eigenfunction $\widetilde{\varphi}$ on the kite such that $\widetilde{\varphi}(\sigma_e(z)) = \widetilde{\varphi}(z)$. Note that whenever we refer to the nodal set of an extended eigenfunction, we are speaking of the nodal set in the extended domain.

If u is an eigenfunction that satisfies Neumann conditions on all three sides, then we will find it useful to reflect about all three sides simultaneously. If some angle of T is greater than $2\pi/3$, then one might not be able to extend u to the union of the three kites unambiguously. But one may use a 'smaller' extension. For example, the bisectors of each angle of the triangle meet at the centroid to form a tripod. This tripod divides the triangle into three smaller triangles each of which contains exactly one edge of T. Reflect each of these smaller triangles about the corresponding edge to obtain a 'hexagon' H_T containing T. The Neumann eigenfunction u on T extends uniquely via the reflection principle to a Laplace eigenfunction $\widetilde{u}$ on H_T.

Let φ be an eigenfunction of the Laplacian on the interior of H_T that extends continuously to the vertices of T. In this article φ will equal $\widetilde{u}$ or $X\widetilde{u}$ where X is either a constant or rotational vector field. By Lemma 2.2, the nodal set of φ is a union of C^1 curves where each curve is either homeomorphic to a circle (a 'loop') or is a proper arc. Recall that each such C^1 curve is called a Cheng curve of φ.

Definition 3.1. Let α be a Cheng curve of φ. The closure of a component of the intersection $\alpha \cap T$ will be called a *maximal subset* of the nodal set of the restriction of φ to T.

The nodal set of the restriction $\varphi|_T$ is a union of maximal subsets. Each maximal subset in the nodal set of φ is either a point, a C^1 loop[5], or a C^1 arc with distinct endpoints in ∂T. Each intersection of such loops/arcs is equiangular (see Remark 2.3). If a maximal subset is homeomorphic to an interval, then we will call it a *maximal arc*.

If a maximal subset consists of a single point, then this point lies in a side of T. Indeed, the nodal set of an eigenfunction defined on an open set has no isolated points. For the same reason, if φ satisfies Neumann or Dirichlet conditions along a side e of T, then e contains no singleton maximal subsets. In particular, the nodal set of the Neumann eigenfunction φ is a union of maximal loops and maximal arcs. Each vertex of the graph $\mathcal{Z}(\varphi)$ is thus either a critical point of φ, an endpoint of a maximal arc, or an isolated point of $\mathcal{Z}(\varphi) \cap \partial T$.

The following should be compared to Lemma 5 in [Sdj15].

Lemma 3.2. *Let T be a triangle. Let φ be an eigenfunction on T that satisfies Neumann conditions along the side e. If a piecewise smooth arc α in $\mathcal{Z}(\varphi)$ has both endpoints in e, then the eigenvalue of φ is strictly greater than the second Neumann eigenvalue of T.*

Proof. The maximal arc and the side e together bound a topological disc D. Define $\widehat{\varphi} : T \to \mathbb{R}$ by setting $\widehat{\varphi}(z) = u(z)$ if $z \in D$ and $\widehat{\varphi}(z) = 0$ otherwise. The H^1 function $\widehat{\varphi}$ satisfies Neumann conditions along e and Dirichlet conditions along the other two sides of T. Hence the eigenvalue, λ, of φ is larger than the first eigenvalue of the mixed eigenvalue problem on T corresponding to Neumann conditions on e and Dirichlet conditions on the other two sides. In turn, by Theorem 3.1 in [Ltr-Rhl17], the first eigenvalue of the mixed problem is greater than the second Neumann eigenvalue of T. □

4. Neumann eigenfunctions on sectors

Let $\Omega \subset \mathbb{C}$ be a sector of angle β and radius $\epsilon > 0$, that is

$$\Omega := \{z : 0 \leq \arg(z) \leq \beta \text{ and } |z| < \epsilon\}.$$

In this section, u is a (real) eigenfunction of the Laplacian on Ω with eigenvalue $\mu > 0$ that satisfies Neumann boundary conditions along the boundary edges corresponding to $\arg(z) = 0, \beta$ respectively. (We impose no conditions on the circle of radius ϵ.) We will use the expansion of u in Bessel functions near the 'vertex' 0, to derive information about both the nodal set and the critical set of u.

Separation of variables leads to the following expansion valid near 0:

$$u\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} c_n \cdot J_{\frac{n\pi}{\beta}}(\sqrt{\mu}\cdot r) \cdot \cos\left(\frac{n\pi\theta}{\beta}\right). \tag{2}$$

Here $c_n \in \mathbb{R}$ and J_ν denotes the Bessel function of the first kind of order ν [Lbv72]. The series converges uniformly on compact sets that miss the origin. The Bessel function J_ν has the expansion [Lbv72]

$$J_\nu(r) = r^\nu \cdot \sum_{k=0}^{\infty} \frac{(-1)^k \cdot r^{2k}}{2^{2k}\cdot\Gamma(k+\nu)\cdot\Gamma(k+\nu+1)}$$

where Γ is the Gamma function. In particular, for each $\nu \geq 0$, there exists an entire function g_ν so that $J_\nu(\sqrt{\mu}\cdot r) = r^\nu \cdot g_\nu(r^2)$.[6] Note that none of the Taylor coefficients of g_ν vanish. In particular, neither g_ν nor g'_ν vanishes in a neighborhood of 0 for each $\nu \geq 0$. With this notation, the expansion in (2) takes a more compact form:

$$u\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} c_n \cdot r^{n\cdot\nu} \cdot g_{n\cdot\nu}\left(r^2\right) \cdot \cos\left(n\cdot\nu\cdot\theta\right) \tag{3}$$

where $\nu = \pi/\beta$.

[5] A loop might not be C^1 at a vertex of T.

[6] Note that though g_ν depends on the eigenvalue μ, we will suppress μ from the notation.

We will be interested in the level set, $u^{-1}(u(0))$, that contains the vertex 0. In particular, if $u(0) = 0$, then $u^{-1}(u(0))$ is the nodal set of u.

Lemma 4.1. *There exists a neighborhood U of 0 such that $u^{-1}(u(0)) \cap U$ either equals $\{0\}$ or equals the union of $m-1$ real-analytic arcs $\alpha_1, \ldots, \alpha_{m-1}$ such that the pairwise intersection of α_j and α_k equals $\{0\}$ for each $j \neq k$.*

Proof. By expanding each g_ν, the expansion in (3) becomes

$$u\left(re^{i\theta}\right) = \sum_{n=0}^{\infty} \sum_{j=0}^{\infty} c_n \cdot a_{n,j} \cdot r^{n\cdot\nu+2j} \cdot \cos\left(n \cdot \nu \cdot \theta\right) \tag{4}$$

where each $a_{n,j}$ is nonzero. We have $u(0) = c_0 \cdot a_0$. Let $\eta = \min\{n \cdot \nu + 2j : (n,j) \neq (0,0), c_n \neq 0\}$ and let $A = \{(n,j) : n \cdot \nu + 2j = \eta\}$. Then

$$\frac{u(z) - u(0)}{r^\eta} = \sum_{(n,j)\in A} c_n \cdot a_{n,j} \cdot \cos(n \cdot \nu \cdot \theta) + h(z) \tag{5}$$

where h is a real-analytic function and both $|h(z)|$ and $|\partial_\theta h(z)|$ are of order $O(|z|^\epsilon)$ as $|z|$ tends to zero for some $\epsilon > 0$. The claim follows from the implicit function theorem. □

We will require more specialized information about the level sets that contain a vertex of a triangle when the vertex angle is acute or obtuse.

Lemma 4.2. *If the angle $\beta < \pi/2$, then there exists a neighborhood U of 0 such that*

1. *if $c_0 \neq 0$, then $U \cap u^{-1}(u(0))$ equals $\{0\}$, and*
2. *if $c_0 = 0$ and $c_1 \neq 0$, then $U \cap u^{-1}(u(0))$ is a simple arc containing 0.*

If $\pi/2 < \beta < \pi$, then there exists a neighborhood U of 0 such that

1. *if $c_1 \neq 0$, then $U \cap u^{-1}(u(0))$ is a simple arc containing 0, and*
2. *if $c_1 = 0$ and $c_0 \neq 0$, then $U \cap u^{-1}(u(0)) = \{0\}$*

If $c_0 = 0 = c_1$, then there exists a neighborhood U of 0 such that $U \cap u^{-1}(u(0))$ consists of at least two arcs.

Proof. Suppose $\beta < \pi/2$. If $c_0 \neq 0$, then η defined in Lemma 4.1 equals 2 and $A = \{(0,1)\}$. In particular, the trigonometric polynomial appearing on the right hand side of (5) is a constant and hence $U \cap u^{-1}(u(0)) = \{0\}$. On the other hand, if $c_0 = 0$ and $c_1 \neq 0$, then $\eta = \nu$ and $A = \{(1,0)\}$. In this case, the trigonometric polynomial of the right hand side of (5) equals $c_1 \cdot a_{1,0} \cdot \cos(\nu \cdot \theta)$, and hence $U \cap u^{-1}(u(0))$ is a simple arc.

Suppose $\pi/2 < \beta < \pi$. If $c_1 \neq 0$, then $\eta = \nu < 2$ and $A = \{(1,0)\}$. Thus, the trigonometric polynomial equals $c_1 \cdot a_{1,0} \cdot \cos(\nu \cdot \theta)$, and $U \cap u^{-1}(u(0))$ is an arc. On the other hand, if $c_1 = 0$ and $c_0 \neq 0$, then the trigonometric polynomial is a constant and hence $U \cap u^{-1}(u(0)) = \{0\}$.

Finally, if $c_0 = 0 = c_1$, then each term in the trigonometric polynomial in (5) is the product of a constant and $\cos(n \cdot \nu \cdot \theta)$ where $n \geq 2$ and $n \cdot \nu + 2j = \eta$. Such a function has at least two roots. □

Proposition 4.3. *If β is not an integer multiple of $\pi/2$, then there exists a deleted neighborhood of $0 \in \Omega$ that contains no critical points of u. If $\beta = \pi/2$, then there exists a neighborhood U of 0 such that $\mathrm{crit}(u) \cap U$ is either empty, equals $\{0\}$, or equals exactly one edge of the sector.*

Remark 4.4. The conditions on β are necessary. For example, on the square $[0,\pi] \times [0,\pi]$ we have the Neumann eigenfunction $u(z) = \cos(x)$. In this case, the set $\{z : x = 0\}$ lies in the critical set of u.

Proof. The point $z = re^{i\theta}$ is a critical point of u if and only if both the radial derivative $\partial_r u$ and the angular derivative $\partial_\theta u$ vanish at z. Let c_m be the first nonzero coefficient in the Bessel expansion in (3). By differentiating term-by-term we obtain

$$\partial_\theta u\left(z\right) = -\sum_{n=m}^{\infty} c_n \cdot r^{n\cdot\nu} \cdot g_{n\cdot\nu}(r^2) \cdot n \cdot \nu \cdot \sin\left(n \cdot \nu \cdot \theta\right) \tag{6}$$

and

$$\partial_r u(z) \;=\; \sum_{n=m}^{\infty} c_n \cdot r^{n\cdot\nu-1} \cdot \left(n\cdot\nu\cdot g_{n\cdot\nu}(r^2) + 2r^2 \cdot g'_{n\cdot\nu}(r^2)\right) \cdot \cos\left(n\cdot\nu\cdot\theta\right). \tag{7}$$

In particular,

$$\partial_\theta u(z) \;=\; -c_m \cdot r^{m\cdot\nu} \cdot g_{m\cdot\nu}(r^2)\cdot m\cdot\nu\cdot \sin\left(m\cdot\nu\cdot\theta\right) \;+\; O\left(r^{(m+1)\nu}\right) \tag{8}$$

and

$$\partial_r u(z) \;=\; c_m \cdot r^{m\cdot\nu-1} \cdot \left(m\cdot\nu\cdot g_{m\cdot\nu}(r^2) + 2r^2\cdot g'_{m\cdot\nu}(r^2)\right)\cdot \cos\left(m\cdot\nu\cdot\theta\right) \;+\; O\left(r^{(m+1)\nu-1}\right) \tag{9}$$

where $O\left(r^k\right)$ represents a function defined in a neighborhood of 0 that is bounded by a constant times r^k.

Suppose that $\partial_\theta u(z) = 0$ and $m \geq 1$. Then since $g_\nu(0) \neq 0$, we find from (8) that $|\sin(m\cdot\nu\cdot\theta)| \;\leq\; O\left(r^\nu\right)$. It follows that there exists $k \in \{0, 1, \ldots, m\}$ so that

$$\left|\theta - \frac{k}{m}\cdot\beta\right| \;=\; O(r^\nu). \tag{10}$$

Suppose that $\partial_r u(z) = 0$. If $m \geq 1$, then $m\cdot\nu \neq 0$, and so from (9) we find that $|\cos(m\cdot\nu\cdot\theta)| < O(r^\nu)$. It follows that there exists $k \in \{0, \ldots, m-1\}$ so that

$$\left|\theta - \frac{2k+1}{2m}\cdot\beta\right| \;=\; O(r^\nu).$$

Therefore, if $m \geq 1$, there exists $\epsilon > 0$ such that if $0 < |z| < \epsilon$ then $\partial_r u(z)$ and $\partial_\theta u(z)$ can not both be zero.

If $m = 0$, then the term associated to $c_m \neq 0$ in (7) might not be dominant and so (9) might not be useful. Which term is dominant depends on the value of $n_0 := \inf\{n \in \mathbb{Z}^+ : c_n \neq 0\}$.

If $\beta < n_0\cdot\pi/2$, then the term associated to c_0 is dominant, and thus $\partial_r u$ does not vanish for small r. If $\beta > n_0\cdot\pi/2$, then the term associated to c_{n_0} is dominant, and we find that there exists $k \in \{0, \ldots, n_0 - 1\}$ so that $|\theta - (2k+1)\cdot\beta/(2n_0)| \;=\; O(r^\epsilon)$ for some $\epsilon > 0$. Comparison with (10) where $m = 0$ then gives that $\partial_\theta u$ and $\partial_r u$ can not both vanish near 0.

If $\beta = \pi/2$, then since u satisfies Neumann conditions along the edges, we may use the reflection principle to extend u to a smooth eigenfunction on the disk $|z| < \epsilon$. By Proposition 2.5, if 0 lies in the critical locus of u, then there exists a disk neighborhood U of zero such that $\mathrm{crit}(u) \cap U = \{0\}$ or $\mathrm{crit}(u) \cap U$ is a real-analytic arc α. Because the extended eigenfunction is invariant under reflection across both the real and imaginary axes, the arc α is also invariant under these reflections and hence lies either in the real or imaginary axis. □

Remark 4.5. If $\beta = \pi$, then the sector is a half-disk. One can apply the reflection principle to extend u to the disk. Using Proposition 2.5, we find that if 0 is a critical point, then there exists a neighborhood U of 0 such that $\mathrm{crit}(u) \cap U$ is either $\{0\}$, equals the real-axis, or is an arc that is orthogonal to the real-axis.

Remark 4.6. If $c_1 \neq 0$, then there exists $r_0 > 0$, such that if $0 < z < r_0$ and $0 < \arg(z) < \beta$, then z is not a critical point of u. Indeed, for each n, $\theta \mapsto \sin(n\cdot\nu\cdot\theta)/\sin(\nu\cdot\theta)$ defines an analytic function on $\mathbb{R}$, and hence from (6) we have

$$\partial_\theta u(z) \;=\; -\sin(\nu\cdot\theta)\cdot r^\nu \left(c_1\cdot\nu\cdot g_\nu(0)\cdot\sin(\nu\cdot\theta) \;+\; O\left(r^{\nu'}\right)\right)$$

where $\nu' = \min\{\nu, 2\}$. Thus, if $z = re^{i\theta}$ is a critical point and $0 < \theta < \beta$ then there exists C such that

$$|c_1|\cdot\nu\cdot g_\nu(0) \;\leq\; C\cdot r^{\nu'}. \tag{11}$$

and therefore $r \geq (|c_1|\cdot\nu\cdot g_\nu(0)/C)^{1/\nu'}$.

5. A second Neumann eigenfunction on a Euclidean triangle

In this section, T is a Euclidean triangle, and u is a second Neumann eigenfunction for T.

We will use the following well-known fact many times in the sequel.

Lemma 5.1. *Let Ω' be a subset of Ω with piecewise smooth boundary, and let $f \in H^1(\Omega)$ that satisfies Dirichlet boundary conditions on Ω', that is $u|_{\partial\Omega'} = 0$. Then the Rayleigh quotient $\mathcal{R}(f) > \mu_2(\Omega)$. In particular, if f itself is a λ-eigenfunction on Ω' with Dirichlet boundary condition, then $\lambda > \mu_2(\Omega)$.*

Proof. By the variational characterization of the first Dirichlet eigenvalue we have $\mathcal{R}(u) \geq \lambda_1(\Omega')$. By the domain monotonicity of the first Dirichlet eigenvalue, $\lambda_1(\Omega') \geq \lambda_1(\Omega)$ and by a result of Polya [Ply52] we have $\lambda_1(\Omega) > \mu_2(\Omega)$, giving the first assertion. ☐

The following fact is also well-known.

Theorem 5.2. *The nodal set of u consists of one simple maximal arc.*

Proof. By Lemma 2.2, the nodal set $\mathcal{Z}(u)$ is a collection of loops and maximal arcs. Lemma 5.1 implies that there are no loops. By Courant's nodal domain theorem, the complement $T \setminus \mathcal{Z}(u)$ has exactly two components. The claim follows. ☐

If v is a vertex of the triangle T, then an ϵ-neighborhood of v can be identified with a subset of a sector. For each vertex v, we consider the Bessel expansion of u about v, and we let c_j^v denote the associated j^{th} Bessel coefficient.

Corollary 5.3. *Let v be a vertex of T. The first two Bessel coefficients, c_0^v and c_1^v, can not both equal zero.*

Proof. If both c_0, c_1 were both zero, then by Lemma 4.2, there would exist (at least) two arcs in $\mathcal{Z}(u)$ that emanate from the vertex. They could not form a loop by Lemma 5.1, and so they would have to be distinct, but this would contradict Theorem 5.2. ☐

Corollary 5.4. *If v and v' are two distinct vertices of T, then $c_0^v = u(v)$ and $c_0^{v'} = u(v')$ can not both equal zero.*

Proof. Suppose to the contrary that c_0^v and $c_0^{v'}$ are both zero. Then by Corollary 5.3, the coefficients c_1^v and $c_1^{v'}$ are both nonzero. Thus, by Lemma 4.2, there would exist an arc in $\mathcal{Z}(u)$ emanating from v and an arc in $\mathcal{Z}(u)$ emanating from v'. By Theorem 5.2 these arcs would belong to the same maximal arc in $\mathcal{Z}(u)$ that joins v and v'. This would contradict Lemma 3.2. ☐

The following is a consequence of a more general result of [Ndr86], but it follows easily from the previous corollary.

Corollary 5.5. *The dimension of the space E of second Neumann eigenfunctions is at most two.*

Proof. Define the linear map $f : E \to \mathbb{R}^2$ by $f(u) = (u(v), u(v'))$ where v and v' are distinct vertices of T. By Corollary 5.4, the map has no kernel, and so the dimension of E is at most two. ☐

Proposition 5.6. *The critical set of a second Neumann eigenfunction u is finite.*

Proof. The Neumann eigenfunction u extends via reflection to an eigenfunction $\widetilde{u}$ on the interior of the 'hexagon' H_T described in §3. By Proposition 2.5, each component of $\text{crit}(\widetilde{u})$ is either an isolated point, a proper analytic arc, or an analytic loop. It follows that each component A of the critical set of u is either an analytic arc with points in the boundary of T, a loop in T, or an isolated point in $T \setminus V$ where V is the set of vertices.

If A were a loop, then each directional derivative, for example $\partial_x u$, would be a Dirichlet eigenfunction on the region bounded by the loop, contradicting Lemma 5.1.

If A were an arc, then the endpoints of the arc lie in the union of two sides. If R is the rotational vector field about the common vertex v of these two sides, then $R_v u$ is a Dirichlet eigenfunction on a subdomain of T. This would contradict Lemma 5.1.

Thus each component of $\mathrm{crit}(u)$ is an isolated point in $T \setminus V$. If each vertex angle is not equal to $\pi/2$, then Proposition 4.3 implies that there is a neighborhood U of the set, V, that contains no critical points. Therefore $\mathrm{crit}(u)$ is finite if T is not a right triangle.

If T is a right triangle, then Proposition 4.3 gives that either a deleted neighborhood of v contains no critical points or one of the sides is a component of $\mathrm{crit}(u)$. In the former case, the preceding argument still applies. The latter case is impossible. Indeed, the other endpoint of the side is a vertex with angle strictly less than $\pi/2$, contradicting Proposition 4.3. □

6. Derivatives of a second Neumann eigenfunction

In this section, u is a second Neumann eigenfunction for a triangle T. Here, we consider the nodal sets of the angular and directional derivatives of u.

By 'directional derivative' we mean the result of applying a (real) constant vector field L. Each such vector field commutes with the Laplacian and so if φ is an eigenfunction of the Laplacian, then $L\varphi$ is also an eigenfunction with the same eigenvalue. We are particularly interested in the unit vector field, L_e, that is parallel to a side e of a triangle T such that a $\pi/2$ counterclockwise rotation of L_e points into the half-plane containing T. We will let $L_e^{\perp}$ denote the unit vector field that is outward normal to the side e. Note that φ satisfies Neumann conditions if and only if $L_e^{\perp}\varphi = 0$ for each side e of T.

By 'angular derivative' we mean the result of applying the rotational vector field R_p that corresponds to the counter-clockwise rotational flow about a point p. To be precise if $p = p_1 + ip_2$, then

$$R_p \;=\; -(y-p_2)\cdot\partial_x \;+\; (x-p_1)\cdot\partial_y.$$

The vector field R_p commutes with the Laplacian, and so if u is a Laplace eigenfunction, then $R_p u$ is also an eigenfunction with the same eigenvalue. We are particularly interested in the case where p is a vertex of a triangle.

Recall that a *tree* is a simply connected graph. The degree of a vertex is the number of edges that contain the vertex. By the *interior of a side* e of the triangle T, we will mean the complement $e \setminus \{v_-, v_+\}$ where v_-, v_+ are the vertices of e. We will denote the interior with e°.

According to §3, the nodal sets of both $R_p u$ and $L_e u$ are locally finite graphs whose vertex set consists of the critical points of u, endpoints of maximal arcs, and isolated points in the boundary of T. We will now show that each of these graphs is finite and each component is a tree.

Lemma 6.1. *Let v be a vertex of T and let e denote the side opposite to v. The nodal set of $R_v u$ is a finite disjoint union of finite trees, and it contains the sides adjacent to v. If the nodal set of $R_v u$ intersects the interior of T, then the nodal set has a degree 1 vertex that lies in the interior of e. Each point that lies in the intersection of e° and $\mathcal{Z}(R_v u)$ is a critical point of u.*

Proof. The simple connectedness of the nodal set $\mathcal{Z}(R_v u)$ follows from Lemma 5.1.

If e' is a side that is adjacent to v, then the restrictions of the vector fields $L_{e'}^{\perp}$ and R_v to e' agree up to a non-zero factor. Thus, since $L_{e'}^{\perp} u$ vanishes along e', so does $R_v u$.

On the other hand, for each z in the side e opposite to v, the vector $L_e^{\perp}(z)$ is independent of the vector $R_v(z)$. Hence, if $z \in \mathcal{Z}(R_v u)$ belongs to the interior of e, then z is a critical point of u. Therefore by Lemma 5.6, the intersection $\mathcal{Z}(R_v u) \cap e$ is finite.

Suppose that z lies in the intersection of $\mathcal{Z}(R_v u)$ and the interior of T. Since $\mathcal{Z}(R_v u)$ is simply connected, the component of $\mathcal{Z}(R_v u)$ that contains z is a maximal arc α that has two distinct endpoints. Since the sides adjacent to v are contained in $\mathcal{Z}(R_v u)$, one of these endpoints lies in the interior of the side opposite to v. This endpoint is a critical point of u.

In sum, the set $\mathcal{Z}(R_v u)$ is the union of the sides adjacent to v and the maximal arcs that have at least one endpoint in the interior of e, and each such endpoint is a critical point of u. Since there are only finitely many critical points and the degree of each vertex of $\mathcal{Z}(R_v u)$ is finite, there are finitely many maximal arcs. It follows that the set $\mathcal{Z}(R_v u)$ is a finite disjoint union of finite trees.

Each finite tree contains at least two degree 1 vertices. Let τ be a (tree) component of $\mathcal{Z}(R_v u)$ that intersects the interior of T. If τ does not contain the union, A, of the sides adjacent to v, then each degree 1 vertex of τ lies in e°. Otherwise, note that the closure of $\tau \setminus A$ is a finite union of trees, and let τ' be a component that intersects the interior. Exactly one vertex of τ' lies in A, and hence at least one degree 1 vertex of τ' lies in e°. □

Corollary 6.2. *If u has a critical point that lies in the interior of T, then for each vertex v of T, the nodal set of $R_v u$ has a degree 1 vertex that lies in the interior of the side opposite to v. In particular, if u has a critical point that lies in the interior of T, then u has at least three more critical points each lying in a distinct side of T.*

Proof. If p is a critical of u, then $R_v u(p) = 0$ for each vertex v of T. Lemma 6.1 implies the claim. □

Lemma 6.3. *Let e be a side of T. The nodal set of $L_e u$ is a finite union of finite trees. If a maximal arc of $\mathcal{Z}(L_e u)$ intersects the interior of T, then one endpoint p of the arc lies in $\partial T \setminus e$, and if p is not the vertex opposite to e, then p is a critical point of u. If the nodal set of $L_e u$ intersects the interior of T, then the nodal set has a degree 1 vertex that lies in $\partial T \setminus e$.*

Proof. Lemma 5.1 implies that the nodal set $\mathcal{Z}(L_e u)$ is simply connected.

Since u satisfies Neumann conditions, the function $L_e u$ satisfies Neumann conditions along e. If $z \in \mathcal{Z}(L_e u) \cap T^\circ$, then there exists a maximal arc of $\mathcal{Z}(L_e u)$ containing z that has distinct endpoints. The endpoints can not both lie in e as a consequence of Lemma 3.2. Hence at least one endpoint lies in $\partial T \setminus e$.

If e' is a side of T that meets e at an angle equal to $\pi/2$, then $L_e = \pm L_{e'}^\perp$. Thus, it follows from Lemma 3.2, that if one of the endpoints of the latter maximal arc lies in e then the other can not lie on e' and hence lies on $\partial T \setminus (e \cup e')$. If e' meets e at an angle not equal to $\pi/2$, then at each $z \in e'$, the vectors $L_e(z)$ and $L_{e'}^\perp(z)$ are independent. In particular z is a critical point of u.

Thus, if α is a maximal arc that intersects the interior of T, then at least one endpoint of α is a critical point that lies in $\partial T \setminus e$. By Lemma 5.6, the set of such points is finite. Each vertex of $\mathcal{Z}(L_e u)$ has finite degree and so the number of maximal arcs in $\mathcal{Z}(L_e u)$ is finite. It follows that $\mathcal{Z}(L_e u)$ is a finite disjoint union of finite trees.

The remainder of the argument is similar to that given at the end of the proof of Lemma 6.1. □

Lemma 6.4. *Let e be a side of T. The intersection $\mathcal{Z}(L_e u) \cap e$ equals the set of critical points of u that lie in e, and is hence finite. Each point in $\mathcal{Z}(L_e u) \cap e$ is an endpoint of at least one maximal arc of $\mathcal{Z}(L_e u)$ that intersects T°.*

Proof. The first assertion follows from the fact that L_e and $L_e^\perp$ are independent. By Lemma 5.6, the function u has only finitely many critical points.

Since u satisfies Neumann conditions along e, we may extend u uniquely to an eigenfunction $\widetilde{u}$ on the interior of the kite K_e that is invariant under the reflection σ_e associated to e. Since L_e is parallel to e, we find that $L_e \widetilde{u}$ is also invariant under σ_e, and hence the nodal set $\mathcal{Z}(L_{e'}(\widetilde{u}))$ is also invariant. No Cheng arc of $\mathcal{Z}(L_{e'}(\widetilde{u}))$ equals e, and therefore there exists a maximal arc that intersects the interior of T. □

The following theorem plays a prominent role in the proof Theorem 1.1.

Theorem 6.5. *Let e be a side of T. If u has a degenerate critical point p that lies in e and u does not have a critical point that lies in the interior of T, then either*

(1) *for each of the vertices v_1, v_2 adjacent to e, the nodal set of $R_{v_i} u$ has a degree 1 vertex that belongs to the edge opposite to v_i, or*

(2) *the nodal set of $L_e u$ has a degree 1 vertex that belongs to the interior of a side e' distinct from e, and the nodal set of $L_{e'} u$ has a degree 1 vertex that belongs to the interior of a side distinct from e'.*

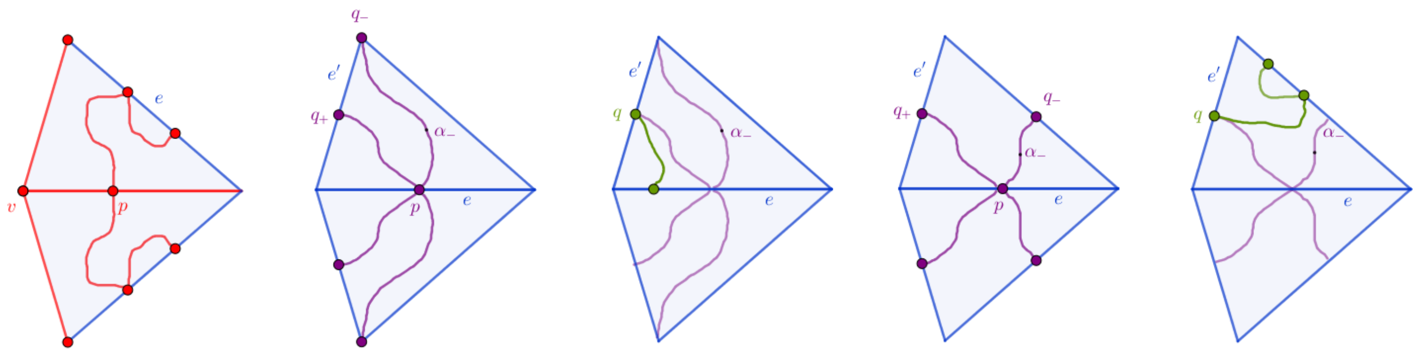

FIGURE 1. Some cases in Theorem 6.5. In the figure on the far left, a possible nodal set of $R_v u$ is described. Possible nodal sets of $L_e u$ are described in the second and fourth pictures from the left, and possible nodal sets of $L_{e'} u$ are described in the third and fifth pictures.

Proof. Without loss of generality, the critical point p is located at the origin, and the edge e lies in the real axis. Let $\widetilde{u}$ be the extension of u to the interior of the kite K_e obtained by reflecting about e. The real-analytic Taylor expansion at $p = 0 + 0i$ has the form

$$\widetilde{u}(z) = a_{00} + a_{20} \cdot x^2 + a_{11} \cdot xy + a_{02} \cdot y^2 + O(3). \tag{12}$$

where $O(k)$ indicates a sum of terms in x and y of order at least k. Since u is a Neumann eigenfunction along e, we have $a_{11} = 0$. Therefore, since by assumption p is a degenerate critical point, either $a_{20} = 0$ or $a_{02} = 0$.

The case $a_{02} = 0$ leads to alternative (1). Indeed, if $a_{02} = 0$, then from (12) we find that the angular derivative of u about an endpoint v of e equals

$$R_v \widetilde{u}(z) = -2a_{20} \cdot xy + O(2) = O(2).$$

It follows that p is a nodal critical point of $R_v \widetilde{u}$. By Lemma 2.2, at least two Cheng curves of $R_v \widetilde{u}$ intersect at p and the intersection is equiangular. In particular, since one of these arcs is e, some other curve is transverse to e at p and hence intersects T°. Hence by Lemma 6.1, the nodal set of $R_v u$ has a degree 1 vertex that lies in the interior of the side opposite to v. (See Figure 1.) Letting $v = v_1, v_2$, the two endpoints of e, we obtain alternative (1).

The case $a_{20} = 0$ leads to alternative (2). Indeed, in this case, from (12) we find that $L_e \widetilde{u} = \partial_x \widetilde{u} = O(2)$. Thus at least two Cheng curves of $\mathcal{Z}(L_e \widetilde{u})$ meet at p. Note that $L_e \widetilde{u}$ is invariant under the reflection about e, and hence these Cheng curves are also invariant. By Lemma 6.4 none of these Cheng curves is a subset of e. It follows that each curve intersects the interior of T. By Lemma 6.3, each of two of the corresponding maximal arcs, $\alpha_\pm$, has an endpoint $q_\pm \in T \setminus e$. If $q_\pm$ is a vertex of T, then the other endpoint, $q_\mp$, lies in the interior of a side e'. If q_+ and q_- lie in the interiors of distinct sides $e'_\pm$, then choose $q = q_+$ and $e' = e'_+$. If q_+ and q_- lie in the interior of the same side e', then by relabeling if necessary, we may assume that q_+ and the vertex v' opposite to e' are separated by α_-, and we choose $q = q_+$. In fact, in each case the curve α_- separates q from the vertex opposite to e'. By Lemma 6.3, q is a critical point of u, and moreover, there exists a degree 1 vertex of $\mathcal{Z}(L_e u)$ that lies in the interior of e'. (See Figure 1.)

Since q is a critical point of u, by Lemma 6.4, there exists a maximal arc α of $\mathcal{Z}(L_{e'} u)$ that intersects the interior of T and has q as an endpoint. The other endpoint of α cannot be the vertex v' opposite to e'. Indeed, since the vectors $L_e(z)$ and $L_{e'}(z)$ are independent at each $z \in T$, an intersection point would be a critical point. By considering a subtree containing the arc α one finds a degree 1 vertex of $\mathcal{Z}(L_{e'} u)$ that lies in the interior of a side distinct from e'. (See Figure 1.) □

Corollary 6.6. *If u has exactly one critical point p, then p is a nondegenerate critical point.*

Proof. By Corollary 6.2, the point p lies in ∂T. By Theorem 6.5, the point p is nondegenerate. □

We will also use the next two results in the proof of Theorem 1.1, but Lemma 6.8 will not be used in the acute case.

Proposition 6.7. *Let e be a side of T. If e contains at least two critical points of u, then u has a third critical point that lies in $\partial T \setminus e$.*

Proof. Let p_+ and p_- be critical points of u that lie on e. By Lemma 6.4, there exists a maximal arc $\alpha_\pm$ of $\mathcal{Z}(L_e u)$ that intersects the interior of T and has $p_\pm$ as an endpoint in e. The other endpoints of $\alpha_\pm$ can not both be equal to the vertex opposite to e, because otherwise we would have a contradiction to Lemma 3.2. The claim then follows from Lemma 6.3. □

Lemma 6.8. *If the first Bessel coefficient c_1 at v equals zero, then u has a critical point lying in the interior of the side that is opposite to v.*

Proof. Let m be the smallest positive integer n such that $c_n \neq 0$. By hypothesis, we have $m \geq 2$, and so from (6) we find that

$$R_v u(z) \;=\; c_m \cdot m \cdot \nu \cdot g_{m\cdot\nu}\left(r^2\right) \cdot r^{m\cdot\nu} \cdot \sin(m \cdot \nu \cdot \theta) \;+\; O\left(r^{(m+1)\cdot\nu}\right).$$

It follows that the nodal set of $R_v u$ intersects the interior, and hence by Lemma 6.1, the edge opposite to v contains a critical point. □

7. One nondegenerate critical point on a side

In this section we show that if a second Neumann eigenfunction u has exactly one critical point, then each vertex is an isolated local extremum of u. To prove this we will consider the 'double' of the triangle and the extension of u to the double.

The construction of the double goes as follows. Let T' be a second triangle in the plane that is isometric to T but disjoint from T, and let $f : T \to T'$ be an isometry. The *double of* T, $DT := T \cup_{f|_{\partial T}} T'$, is the topological space obtained by identifying ∂T to $\partial T'$ via the restriction of f. The space DT is homeomorphic to a 2-dimensional sphere. The space DT has three 'cone points', Dv_1, Dv_2, Dv_3, each corresponding to a vertex of T. The complement of these cone points has a smooth Riemannian metric whose restriction to T and T' coincides with the Euclidean metric.

Define $Du : DT \to \mathbb{R}$ by setting $Du(z) = u(z)$ if $z \in T$ and $Du(z) = u \circ f^{-1}(z)$ if $z \in T'$. The restriction of Du to $DT \setminus \{Dv_1, Dv_2, Dv_3\}$ is a smooth Laplace eigenfunction.

Let $\chi(X)$ denote the Euler characteristic of a cell complex, that is, $\chi(X)$ is the alternating sum of the number of k-cells. Each surface (resp. graph) is a cell complex, and the Euler characteristic only depends on the topology of the surface (resp. graph).

Recall that the Morse index of a nondegenerate critical point is the sum of the dimensions of the eigenspaces of the Hessian that have have negative eigenvalues. In particular, the Morse index of a nondegenerate critical point of a smooth function defined on a surface equals 1 iff the determinant of the Hessian is negative. We will say that a point p is an *isolated local extrema* of u provided the connected component of $u^{-1}(u(p))$ that contains p equals $\{p\}$.

Proposition 7.1. *If u has exactly one critical point p, then each vertex v is an isolated local extremum of u and the critical point is nondegenerate with Morse index 1. If u has no critical points, then exactly two of the vertices are isolated local extrema.*

Proof. Consider the level sets of $Du : DT \to \mathbb{R}$. Let A be the union of the level sets that contain a critical point of u or a vertex of T. The complement $S := DT \setminus A$ is foliated by the levels sets of Du that are each homeomorphic to a circle. In particular, the set $DT \setminus A$ is a disjoint union of annuli. Since each annulus in DT is obtained by adding a single 1-cell and a single 2-cell to A, the surface DT is obtained from the graph A by adding the same number of 1-cells and 2-cells. It follows that $2 = \chi(DT) = \chi(A)$.

By Lemma 4.2 and Corollary 5.3, the component of a level set of u that contains a vertex v either equals $\{v\}$ or is a simple arc. Thus, the the component of a level set of Du that contains a cone point either consists of the cone point or is a simple loop. Note that each isolated point has Euler charateristic equal to one, and each loop has Euler characteristic equal to zero.

Suppose that u has exactly one critical point, p. By Corollary 6.6, the critical point p is nondegenerate and belongs to a side of T. The level set, Γ, of Du that contains Dp either equals $\{Dp\}$ or is homeomorphic to a figure eight. The Euler characteristic of the figure eight is -1, and so $2 = \chi(A) = \pm 1 + k$ where k is the number of vertices that are isolated extrema of u. Thus, $k = 1$ or $k = 3$. The case $k = 1$ is impossible by Corollary 5.4. Hence Γ is a figure eight, and it follows that the Morse index at p equals 1.

If u has no critical points, then $2 = \chi(A) = k$, and so exactly two vertices are isolated local extrema of u. ☐

8. The behavior of critical points along a path of triangles

In this section we consider the behavior of second Neumann eigenfunctions associated to a one parameter family of labeled triangles.

Let (v_1, v_2, v_3) be the labeled vertices of a non-equilateral, non-right triangle. Define the 'straight line' path[7] to the right isosceles triangle with vertices $(0, 1, i)$ by

$$(v_1(t), v_2(t), v_3(t)) \ := \ (1-t)\cdot(v_1, v_2, v_3) \ + \ t\cdot(0, 1, i). \tag{13}$$

Let T_t denote the triangle with vertices $(v_1(t), v_2(t), v_3(t))$. By relabeling the vertices of (v_1, v_2, v_3) if necessary, we may assume that the angle at v_1 is greater than $\pi/3$. If T_0 is acute, then for each $t < 1$, the triangle T_t is acute and not equal to the equilateral triangle. If T_0 is obtuse, then for each $t < 1$, the triangle T_t is obtuse and hence can not be the equilateral triangle. By the results of [Sdj15], [Mym13], and [Atr-Brd04], the second Neumann eigenvalue of T_t is simple for each $t \in [0, 1]$.[8] Let h_t be the unique real affine homeomorphism that maps the ordered triple $(0, 1, i)$ to $(v_1(t), v_2(t), v_3(t))$. Standard perturbation theory implies that the second Neumann eigenvalue $\mu_2(t)$ of the triangle T_t varies continuously with t and for each t there exists a $\mu_2(t)$-eigenfunction, $u_t : T_t \to \mathbb{R}$, such that $t \mapsto u_t \circ h_t$ is continuous.[9]

Let $v = v_i(t)$ be one of the vertices, and consider the 'Bessel expansion' about v as in (3):

$$u_t\left(re^{i\theta}\right) \ = \ \sum_{n=0}^{\infty} c_n(t)\cdot r^{n\cdot\nu_t}\cdot g^t_{n\cdot\nu_t}\left(r^2\right)\cdot\cos\left(n\cdot\nu_t\cdot\theta\right) \tag{14}$$

where $\nu_t = \pi/\beta_t$ and β_t is the angle at $v(t)$. Because the functions $t \mapsto \mu_2(t)$ and $t \mapsto u_t \circ h_t$ are both continuous, each quantity in (14) depends continuously on t.

Proposition 8.1. *Let t_n converge to $t \le 1$, and for each n, let p_n be a critical point of u_{t_n} that lies in the interior of the triangle T_{t_n}. Then the sequence p_n can not converge to a vertex $v_i(t)$.*

Proof. Suppose to the contrary that p_n converges to a vertex $v = v_i(t)$. Since the angle β at v is less than π, we have $\nu_t > 1$. In particular ν_{t_n} is uniformly bounded from below by some $\overline{\nu} > 1$. From the Bessel expansion (14), we find that the radial derivative of u_s satisfies

$$\partial_r u_s(z) \ = \ 2c_0(s)\cdot r\cdot g_0'(0) \ + \ c_1(s)\cdot\nu_s\cdot g_{\nu_s}(0)\cdot r^{\nu_s-1}\cdot\cos(\nu_s\cdot\theta) \ + \ O\left(r^{\nu_s^*}\right) \tag{15}$$

where $\nu_s^* = \min\{3, 2\nu_s - 1, \nu_s + 1\}$ and the remainder term depends continuously on s. Since $p_n = r_n \cdot e^{i\theta_n}$ is a critical point, we have $\partial_r u_{t_n}(p_n) = 0$. Thus, from (15) we find that there exist a constant $\nu^* > 1$ and a

[7] Our methods apply provided the path is continuous and, for each $t < 1$, the triangle has no right angle and is not equilateral.

[8] In fact, one can avoid using the simplicity of μ_2 by making some additional arguments (see §11). Also, note that when we first considered this question, the foreknowledge that μ_2 is simple made our approach seem more feasible.

[9] The function u_t lies in $C^0(T_t)$ and in the complement of the vertices $T_t \setminus \{v_1(t), v_2(t), v_3(t)\}$ it lies in the Sobolev space H^s for each s, and hence in C^k for each k. Continuity takes place in these spaces.

constant $C > 0$ so that for each n

$$\left|2c_0(t_n)\cdot g_0'(0) \;+\; c_1(t_n)\cdot \nu \cdot g_{\nu_{t_n}}(0)\cdot r_n^{\nu_{t_n}-2}\cdot \cos(\nu_n\cdot\theta_n)\right| \;\le\; C\cdot (r_n)^{\nu^*-1}. \tag{16}$$

On the other hand, from (11) we obtain a constant $C' > 0$ such that for each n

$$|c_1(t_n)|\cdot \nu_n \cdot g_{\nu_n}(0) \;\le\; C'\cdot r_n.$$

By combining this with (16), we obtain

$$|2c_0(t_n)\cdot g_0'(0)| \;\le\; (C+C')\cdot (r_n)^{\nu^*-1}.$$

Hence $c_0(t) = 0 = c_1(t)$, but this contradicts Lemma 5.3. □

Lemma 8.2. *Let t_n converge to $t \le 1$, and suppose that for each n, the points p_n and q_n are critical points of u_{t_n} that lie in the boundary of the triangle. Suppose that p_n converges to a vertex v and q_n converges to a vertex v'. If for each n, the points p_n and q_n lie in distinct sides, then $v \ne v'$.*

Proof. Suppose to the contrary that $v = v'$, and consider the Bessel expansion of u_{t_n} about this vertex. Let β_n be the angle at v, and let $\nu_n = \pi/\beta_n$. By hypothesis, for each n we have either $\arg(p_n) = 0$ and $\arg(q_n) = \beta_n$ or $\arg(p_n) = \beta_n$ and $\arg(q_n) = 0$. Thus, since p_n and q_n are both critical points, we find from (14) that

$$0 \;=\; 2c_0(t_n)\cdot|p_n|\cdot g_0'\left(|p_n|^2\right) \;+\; c_1(t_n)\cdot\nu_n\cdot g_{\nu_n}\left(|p_n|^2\right)\cdot|p_n|^{\nu_n-1} \;+\; O\left(|p_n|^{\nu_n+1}\right) \;+\; O\left(|p_n|^{2\nu_n-1}\right) \tag{17}$$

and

$$0 \;=\; 2c_0(t_n)\cdot|q_n|\cdot g_0'\left(|q_n|^2\right) \;-\; c_1(t_n)\cdot\nu_n\cdot g_{\nu_n}\left(|q_n|^2\right)\cdot|q_n|^{\nu_n-1} \;+\; O\left(|q_n|^{\nu_n+1}\right) \;+\; O\left(|q_n|^{2\nu_n-1}\right). \tag{18}$$

Divide (17) by $|p_n|\cdot g_0'\left(|p_n|^2\right)$ and (18) by $|q_n|\cdot g_0'\left(|q_n|^2\right)$, and subtract the resulting equations to find that

$$0 \;=\; c_1(t_n)\cdot\nu_n\cdot\left(A_n\cdot|p_n|^{\nu_n-2} \;+\; B_n\cdot|q_n|^{\nu_n-2}\right) \;+\; O\left(|p_n|^{\nu_n^*} \;+\; |q_n|^{\nu_n^*}\right), \tag{19}$$

where $A_n = g_{\nu_n}(|p_n|)/g_0'(|p_n|^2)$, where $B_n = g_{\nu_n}(|q_n|)/g_0'(|q_n|^2)$, and where $\nu_n^* = \min\{2\nu-2, \nu_n\}$. Since $A_n\cdot B_n$ converges to $(g_{\nu_n}(0)/g_0'(0))^2$, for sufficiently large n we have $A_n\cdot B_n > 0$. Also note that $\nu_n^* > \nu_n - 2$. In general, if $a, b > 0$ and $x, y \in \mathbb{R}$, then $(a^x+b^x)/(a^y+b^y) \le a^{x-y}+b^{x-y}$. Since A_n and B_n are bounded sequences, we may apply this inequality to (19) and find that $c_1(t_n) = O(|p_n|^{\nu_n^*-\nu_n+2}+|q_n|^{\nu_n^*-\nu_n+2})$. Therefore, as n tends to infinity, the sequence $c_1(t_n)$ tends to zero.

Divide (17) by $|p_n|^{\nu_n-1}\cdot g_{\nu_n}\left(|p_n|^2\right)$ and (18) by $|q_n|^{\nu_n-1}\cdot g_{\nu_n}\left(|q_n|^2\right)$, and add the resulting equations to find that

$$0 \;=\; 2c_0(t_n)\cdot\left(A_n^{-1}\cdot|p_n|^{2-\nu_n} \;+\; B_n^{-1}\cdot|q_n|^{2-\nu_n}\right) \;+\; O\left(|p_n|^{\nu_n'} \;+\; |q_n|^{\nu_n'}\right),$$

where $\nu_n' = \min\{2, \nu_n\}$. Because $\beta < \pi$, we have that $\nu_n' > 2-\nu_n$. Since A_n^{-1} and B_n^{-1} are bounded sequences, it follows that $c_0(t_n) = O(|p_n|^{\nu_n'-2+\nu_n}+|q_n|^{\nu_n'-2+\nu_n})$. In particular $c_0(t_n)$ converges to zero.

So we have shown that $c_0(t) = 0 = c_1(t)$, but this contradicts Corollary 5.3. □

Lemma 8.3. *Let t_n converge to $t \le 1$. Suppose that for each n, the point p_n is a critical point of u_{t_n} and p_n converges to a vertex v. If the limiting angle β at v is less than $\pi/2$, then $u_t(v) = 0$. If $\pi/2 < \beta < \pi$, then the first Bessel coefficient, c_1, of u_t at v equals 0.*

Proof. By Proposition 8.1 and passing to a subsequence if necessary, we may assume, without loss of generality, that each p_n lies in a side e of T_{t_n}. Let β_n be the angle at the vertex v of T_{t_n}. For each n, consider the Bessel expansion of u in the sector with vertex v so that e corresponds to $\theta = 0$. From (7) we find that

$$0 \;=\; \partial_r u(p_n) \;=\; 2c_0(n)\cdot|p_n|\cdot g_0'(0) \;+\; c_1(n)\cdot\nu\cdot|p_n|^{\nu_n-1}\cdot g_\nu(0) \;+\; O\left(|p_n|^3\right) + \;O\left(|p_n|^{\nu+1}\right) \tag{20}$$

where $c_0(n)$ (resp. $c_1(n)$) is the zeroth (resp. first) Bessel coefficient of u_{t_n} at v, where $\nu_n = \pi/\beta_n$. We have $\beta = \lim\beta_n$, and we let $\nu = \lim\nu_n$.

If $\beta < \pi/2$, then there exists $N > 0$ so that if $n > N$, then $\nu_n - 2 \ge (\nu-2)/2 := \epsilon > 0$. Thus, from (20) we have

$$0 \;=\; 2c_0(n)\cdot g_0'(0) \;+\; O\left(|p_n|^\epsilon\right) \;+\; O\left(|p_n|^2\right),$$

and hence $c_0(n)$ converges to zero. Thus, since u_{t_n} converges to u_t, the zeroth Bessel coefficient of u_t at v equals zero. Hence $u_t(0) = 0$.

If $\pi/2 < \beta < \pi$, then there exists $N > 0$ so that if $n > N$, then $2 - \nu_n \geq (2 - \nu_n)/2 := \epsilon > 0$. Thus, from (20) we have

$$0 = c_1(n) \cdot \nu_n \cdot g_{\nu_n}(0) + O\left(|p_n|^{\epsilon}\right) + O\left(|p_n|^2\right),$$

and hence $c_1(n)$ converges to zero. Thus, since u_{t_n} converges to u_t, first Bessel coefficient of u_t at v equals zero. □

9. On the stability of critical points

In this section, we study the behavior of the critical points of u_t as t varies. In particular, we prove some results about the 'stability' of critical points of a continuous family of eigenfunctions $s \mapsto \varphi_s$ defined on an open domain $\Omega \subset \mathbb{C}$. We say that a critical point p of φ_t is *stable* iff for each neighborhood U of p, there exists $\epsilon > 0$ so that if $|s - t| < \epsilon$, then U contains a critical point of u_s. Our first lemma is more or less standard; it says that non-degenerate critical points are stable.

Lemma 9.1. *Let $\Omega \subset \mathbb{C}$ be an open set, and, for each $s \in (-\delta, \delta)$, let $\varphi_s : \Omega \to \mathbb{R}$ be an eigenfunction of the Laplacian such that $s \mapsto \varphi_s$ in continuous. If $p \in \Omega$ is a nondegenerate critical point of φ_0, then there exist $\epsilon > 0$ and a path $p : (-\epsilon, \epsilon) \to \Omega$ such that $p(0) = p$ and $p(s)$ is a nondegenerate critical point of φ_s for each $s \in (-\epsilon, \epsilon)$.*

Proof. The Hessian of φ_0 at p has two nonzero (real) eigenvalues. Thus, there exists a real-affine map $h : \mathbb{C} \to \mathbb{C}$ so that $h(0) = p$ and the Taylor expansion of $\varphi_0 \circ h$ at 0 has the form

$$\varphi_0 \circ h(z) = \varphi_0(p) + x^2 \pm y^2 + O(3)$$

where $O(3)$ is a function such that vanishes to order 3 in x, y. By the continuity of $s \mapsto \varphi_s$, this expansion extends to s near t:

$$\varphi_s \circ h(z) = \varphi_s(p) + a_{10}(s) \cdot x + a_{01}(s) \cdot y + a_{20}(s) x^2 + a_{11}(s) \cdot xy + a_{02}(s) \cdot y^2 + O(3).$$

We have

$$\begin{aligned} \partial_x^2 \varphi_s \circ h(z) &= 2a_{20}(s) + O(1) \\ \partial_y^2 \varphi_s \circ h(z) &= 2a_{02}(s) + O(1) \end{aligned}$$

where $a_{20}(0) = 1$ and $a_{02}(0) = \pm 1$. Since a_{jk} is continuous in s, there exists $\epsilon > 0$ so that if $|s| < \epsilon$, then $a_{20}(s) > 1/2$ and $|a_{02}(s)| > 1/2$. It follows from the implicit function theorem that there exists a neighborhood U of 0 such that for each $|s| < \epsilon$, the intersection $\mathcal{Z}(\partial_x \varphi_s \circ h) \cap U$ (resp. $\mathcal{Z}(\partial_y \varphi_s \circ h) \cap U$) is a real-analytic arc α_s (resp. β_s) that depends continuously on s. The arcs α_0 and β_0 intersect transversely at the origin, and hence there exists a neighborhood $U' \subset U$ of 0 and $\epsilon' > 0$ so that if $|s - t| < \epsilon'$, then α_s and β_s have a unique intersection point $p(s) \in U'$, and moreover, p_s depends continuously on s. In particular, if $|s| < \epsilon'$, then φ_s has no critical points in U' other than $p(s)$. □

Now we apply the above lemma to our path u_t of second Neumann eigenfunctions.

Corollary 9.2. *If $p_t \in T_t$ is a nondegenerate critical point of u_t, then there exists $\epsilon > 0$ and a path $p : (t - \epsilon, t + \epsilon) \to \mathbb{C}$ such that $p(t) = p_t$ and $p(s)$ is a nondegenerate critical point of u_s that lies in T_s. Moreover, if p_t lies in the boundary of T_t, then p_s lies in the boundary of T_s for each $s \in (t - \epsilon, t + \epsilon)$.*

Proof. If p_t lies in the interior of T_t, then there exist $\epsilon' > 0$ and an open ball B about p_t so that if $|s - t| < \epsilon$, then $B \subset T_s^{\circ}$. By applying Lemma 9.1 to the restriction of $t \mapsto u_t$ to B, we find the desired $\epsilon < \epsilon'$ and path $s \mapsto p(s)$.

If p_t lies in the boundary of T_t, then p_t lies on the interior of a side e_t of T_t. Let K_e^s denote the kite obtained from reflecting T_s across e_s with the reflection σ_e^s. Since u_s is a Neumann eigenfunction, we may extend u_s via reflection to an eigenfunction $\widetilde{u}_s$ defined on K_e^s for each $s \in (t-\epsilon, t+\epsilon)$. Note that there exist $\epsilon' > 0$ and an open ball B about p_t so that if $|s-t| < \epsilon'$, then B lies inside K_e^s.

The point p_t is a nondegenerate critical point of $\widetilde{u}_t$, and hence Lemma 9.1 implies that there exists $0 < \epsilon < \epsilon'$ and a path $p(s) : (t-\epsilon, t+\epsilon) \to B$ so that $p(t) = p_t$ and so that $p(s)$ is a nondegenerate critical point of $\widetilde{u}_s$ for each $s \in (t-\epsilon, t+\epsilon)$.

Suppose that for some s, the critical point $p(s)$ does not lie in the side e_s. Then $\sigma_e^s(p(s))$ would be a distinct critical point of $\widetilde{u}_s$. On the other hand, we have $\sigma_e^t(p(t)) = p(t)$, and so we would find an s' such that $p(s')$ is a degenerate critical point, a contradiction. Therefore, $p(s)$ lies in e_s for each $s \in (t-\epsilon, t+\epsilon)$. ☐

Let $H = \{z \in \mathbb{C} : \mathrm{Im}(z) \geq 0\}$ denote the closed upper half plane.

Lemma 9.3. *Let $\Omega \subset \mathbb{C}$ be an open set that contains 0, and, for each $s \in (-\delta, \delta)$, let $\varphi_s : \Omega \to \mathbb{R}$ be an eigenfunction of the Laplacian such that $s \mapsto \varphi_s$ in continuous. Suppose that the intersection $\mathcal{Z}(\varphi_0) \cap H \cap \Omega$ consists of a simple arc α whose intersection with the real axis is the point 0. Then there exists $\epsilon > 0$ so that if $|s| < \epsilon$, then $\mathcal{Z}(\varphi_s)$ intersects the real axis.*

Proof. It follows from Lemma 2.2 and Lemma 2.4 that there exists a circle $C = \{z \in \mathbb{C} : |z| = r\}$ so that $\mathcal{Z}(\varphi_0) \cap C$ is finite, each intersection is transverse, and $\mathcal{Z}(\varphi_0) \cap C \cap H$ contains exactly one point z_0. Thus, since $s \to \phi_s$ is continuous, there exists $\epsilon > 0$ so that each intersection point $z_s \in \mathcal{Z}(\varphi_s) \cap C$ depends continuously on $s \in (-\epsilon, \epsilon)$ and each intersection is transverse. Since $s \mapsto z_s$ is continuous, we may assume, by choosing $\epsilon > 0$ smaller if necessary, that if $|s-t| < \epsilon$ then $\mathcal{Z}(\varphi_0) \cap C \cap H$ consists of exactly one point z_s^0.

Let α_s be the (proper) Cheng curve of $\mathcal{Z}(\varphi)$ that contains z_s^0. Since the intersection of α_s and C is transverse, the arc α_s intersects the open disc $\{z \in \mathbb{C} : |z| < r\}$. Since α_s^0 is a proper curve in Ω, the curve α_s intersects C at a point z_s^1 distinct from z_s^0. The point z_s^1 lies outside of H. Therefore, by, for example, the intermediate value theorem, the Cheng curve α_{z_s} intersects the real axis. ☐

Proposition 9.4. *If u_t has a critical point that lies in the interior of the triangle T_t, then there exists $\epsilon > 0$ such that if $|t-s| < \epsilon$, then u_s has at least three critical points.*

Proof. Corollary 6.2 implies that for each vertex v of the triangle, the nodal set of $R_v u$ has a degree 1 vertex p in the side e that is opposite to v. For each s, let $\widetilde{u}_s$ denote the extension of u to the kite $K_{e,s}$ obtained by reflecting about the side e. There exists $\epsilon_1 > 0$ and a neighborhood Ω of p so that if $|s-t| < \epsilon_1$, then Ω lies in the interior of $K_{e,s}$. For such s, consider the restriction of $\widetilde{u}_s$ to U. Without loss of generality, $p = 0$ and e lies in the real axis, and so we may apply Lemma 9.3 to find $\epsilon > 0$ so that if $|s-t| < \epsilon$, then $\mathcal{Z}(R_v \widetilde{u}_s) \cap e$ contains a point p_s in the interior of e. Since the vectors $L_e^{\perp}(p_s)$ and $R_v(p_s)$ are independent, the point p_s is a critical point of u_s. Thus, for each vertex v of T_s and $s \in (t-\epsilon, t+\epsilon)$, there exists a critical point of u that lies in the interior of the side opposite to v. ☐

Proposition 9.5. *If u_t has a degenerate critical point p, then there exists $\epsilon > 0$ such that if $|t-s| < \epsilon$, then u_s has at least two critical points.*

Proof. If p lies in the interior of T_s, then this follows from Proposition 9.4.

If p lies in a side, then Theorem 6.5 provides two cases, and in each case we have two eigenfunctions on the triangle T_t each of whose nodal sets has a degree 1 vertex. By applying Lemma 9.3 in the same manner as it was applied in the proof of Proposition 9.4, we find that the nodal sets of the perturbed eigenfunctions intersect the relevant sides giving critical points for each s near t. ☐

10. The proof of Theorem 1.1

Let $t \mapsto u_t$ be the one-parameter family of eigenfunctions defined at the beginning of §8. For each $t \in [0,1]$, let $N(t)$ denote the number of critical points of u_t.

Lemma 10.1. *Suppose that T_0 is an acute triangle. If $N(0) \geq 2$, then $N(t) \geq 2$ for each $t < 1$.*

Proof. It suffices to show that the set $\{t \in [0,1) : N(t) \geq 2\}$ is both open and closed in $[0,1)$.

(Open) Suppose $t < 1$ and $N(t) \geq 2$. If u_t has an interior critical point, then Proposition 9.4 implies that $N(s) \geq 2$ for each s in a neighborhood of t. If u_t has a degenerate critical point on a side, then Proposition 9.5 implies that $N(s) \geq 2$ for each s in a neighborhood of t. If each critical point of u_t is nondegenerate, then it follows from Corollary 9.2 that $N \geq 2$ in a neighborhood of t.

(Closed) Let t_n converge to $t < 1$ and suppose that $N(t_n) \geq 2$ for each n. Let p_n and q_n be distinct critical points of u_{t_n}. Suppose that a subsequence of p_n converges to a vertex v, and a subsequence of q_n converges to a vertex v'. Abusing notation slightly, we denote the subsequences with p_n and q_n. Lemma 8.1 implies that, by passing to a further subsequence if necessary, we may assume that p_n and q_n lie in the boundary of the triangle, and hence, by Proposition 6.7, they lie in distinct sides. Lemma 8.2 gives that $v \neq v'$. Since the limiting angles at v and v' are both less than $\pi/2$, Lemma 8.3 implies that $u(v) = 0$ and $u(v') = 0$. This contradicts Corollary 5.4.

Therefore, at most one of the sequences, p_n, q_n, has a vertex as an accumulation point. Suppose, without loss of generality, that p_n has an accumulation point p that is not a vertex. Since u_{t_n} converges to u_t, the point p is a critical point of u_t, and so $N(t) \geq 1$.

If q_n has an accumulation point q that is not a vertex, then q is also a critical point u_t. If $p \neq q$, then $N(t) \geq 2$. If $p = q$, then p is a degenerate critical point, and hence and hence $N(t) \geq 2$ by Proposition 9.5.

If q_n has a subsequence that converges to a vertex v, then it follows from Lemma 8.3 that $u(v) = 0$. Thus, by Lemma 4.2, the vertex v is not an isolated local extremum. Therefore, Proposition 7.1 implies that $N(t) \neq 1$, and so $N(t) \geq 2$. □

Next, we consider the case where T_t is an obtuse triangle for each $t < 1$. Let $v_o(t)$ denote the vertex of T_t whose angle is greater than $\pi/2$. Let $c_1(t)$ denote the first Bessel coefficient of u_t at $v_o(t)$.

Lemma 10.2. *Suppose that T_0 is an obtuse triangle. If $N(0) \geq 2$, then $N(t) \geq 1$ for each $t < 1$. If $t < 1$ and $N(t) = 1$, then $c_1(t) = 0$.*

Proof. Let $A := \{t \in [0,1) : N(t) \geq 2\}$ and let $B := \{t \in [0,1) : c_1(t) = 0 \text{ and } N(t) = 1\}$. To prove the Lemma, it suffices to show that $A \cup B$ is both open and closed in $[0,1)$.

(Open) The set A is open by the same argument given in the proof of Lemma 10.1. If $t \in B$, then u_t has exactly one critical point and it is non-degenerate by Corollary 6.6. Thus, by Corollary 9.2, there exists $\epsilon > 0$ such that if $0 < |s - t| < \epsilon$, then $N(s) \geq 1$. If $|s - t| < \epsilon$ and $c_1(s) = 0$, then $s \in B$. If $|s - t| < \epsilon$ and $c_1(s) \neq 0$, then Lemma 4.2 implies that the vertex v_0 is not an isolated local extrema of u_s, and hence, by Proposition 7.1, we have $N(s) \geq 2$. Thus, $s \in A$.

(Closed) The set B is closed. Let $t_n \in A$ be a sequence that converges to $t < 1$. Let p_n and q_n be distinct critical points of u_{t_n}. By Proposition 6.7, if p_n and q_n both lie in the boundary of the triangle, then we may assume that they lie in distinct sides.

Suppose that neither p_n nor q_n has a vertex as an accumulation point. Let p (resp. q) be an accumulation point of p_n (resp. q_n). Since the points p and q lie on different sides, $p \neq q$, and hence $N(t) \geq 2$ implying that $t \in A$.

Now suppose that a vertex v is an accumulation point of p_n, and suppose that v' is an accumulation point of q_n. We may argue as in the proof of Lemma 10.1 to show that it is not possible that both of the angles at v and v' are less than $\pi/2$. Thus, without loss of generality, the vertex v equals the vertex $v_o(t)$ whose angle is greater than $\pi/2$. In particular, by Lemma 8.3, we have $c_1(t) = 0$, and hence, by Lemma 6.8, we have $N(t) \geq 1$. Thus, $t \in A \cup B$.

Finally, suppose that p_n has an accumulation point which equals a vertex v, whereas q_n has an accumulation point, q, which is not a vertex. The point q is a critical point and so $N(t) \geq 1$. If $v = v_0$, then as before $c_1(t) = 0$, and hence $t \in A \cup B$. If $v \neq v_0$, then by Lemma 8.3, we have $u(v) = 0$. Thus, Lemma 4.2 implies that v is not an isolated local extremum. It follows from Proposition 7.1 that $N(t) \geq 2$, and hence $t \in A$. □

We will show that Lemmas 10.1 and 10.2 imply that if u_0 has at least two critical points, then u_t has at least two critical points for each $t < 1$ and sufficiently close to 1. In contrast, the function u_1 has no critical points.[10] Indeed, each eigenfunction for the right isosceles triangle $(0,1,i)$ is a multiple of the function

$$u(z) = \cos(\pi x) - \cos(\pi y) \tag{21}$$

where as usual $z = x + iy$.

Proof of Theorem 1.1. We assume that u_0 has an interior critical point and derive a contradiction. If u_0 has an interior critical point, then Corollary 6.2 implies that $N(0) \geq 2$. In the acute case, Lemma 10.1 shows that there exists a sequence t_n converging to 1 such that $N(t_n) \geq 2$ for each n. In the obtuse case, observe that μ_1 is simple and u_1 is a multiple of the function u in (21). This implies that for $t < 1$ and sufficiently close to 1, the eigenvalue μ_t is simple and $c_1(t) \neq 0$. Hence by the last part of Lemma 10.2 there exists a sequence t_n converging to 1 such that $N(t_n) \geq 2$ for each n.

Let p be an accumulation point of a sequence of critical points, p_n, of u_{t_n}. If p is not a vertex of the triangle T_1, then p is a critical point of u_1. But u_1 is a multiple of the function u described in (21), and u has no critical points. Therefore, each accumulation point of $\{p_n\}$ is a vertex. It follows from Proposition 8.1 that there exists $K > 0$ such that if $n > K$, then each critical point of u_{t_n} lies in a side of the triangle.

Since $N(t_n) \geq 2$, Proposition 6.7 implies that that for each $n > K$ there exist distinct sides e and e' and sequences of critical points p_n and p'_n so that for each n we have $p_n \in e$ and $p'_n \in e'$. By passing to a subsequence if necessary we may assume that p_n converges to a vertex $v = v_j(1)$ and p'_n converges to a vertex $v' = v_k(1)$. By Lemma 8.2, we have $v \neq v'$. The sets $\{v, v'\}$ and $\{1, i\}$ are both contained in a three element set, and hence we may assume without loss of generality that $v = 1$ or $v = i$. Thus, by Lemma 8.3, the function u_1 vanishes at either 1 or i. But u_1 is a multiple of the function u described in (21), and u does not vanish at 1 or i.

We have thus proven that if T_0 is either an obtuse or a non-equilateral acute triangle, then a second Neumann eigenfunction for T_0 has at most one critical point, and if such a critical point exists, then it lies in ∂T_0. Now we use this to prove the claim for right triangles. (The case of the equilateral triangle can be established by direct computation [Lmé52] [Pns80].)

Given a (labeled) right triangle T, let $t \mapsto T_t$ be a path of labeled triangles such that $T_0 = 0$ and if $t \neq 0$, then T_t is an acute triangle (and not equilateral). Let u be an eigenfunction on T_0 associated to $\mu_2(T)$. For each t, the eigenvalue $\mu_2(T_t)$ is simple [Sdj15], and hence standard perturbation theory implies that there exists a continuous path $t \mapsto u_t$ of eigenfunctions associated to $\mu_2(T_t)$ such that $u_0 = u$. If u were to have an interior critical point, then a variant of Proposition 8.1, would imply that u_t has at least three critical points for small t. If u were to have a degenerate critical point that belonged to a side of T, then a variant of Proposition 9.5 would imply that u_t would have at least two critical points for t small. Finally, if u were to have more than one nondegenerate critical point, then Lemma 9.1 would imply that u_t would have at least two critical points for t small. Each is a contradiction. □

11. Working without the assumption of simplicity

In this section, we indicate the modifications needed to avoid using the simplicity of μ_2 for non-equilateral triangles. Our discussion begins with a standard application of analytic perturbation theory [Kato].

Lemma 11.1. *For each $i \in \mathbb{N}$, there exist an analytic path*[11] *$t \mapsto \varphi_i(t)$ and an analytic path $t \mapsto \lambda_i(t)$ so that for each $t \in [0,1]$, the function $\varphi_i(t)$ is a Neumann eigenfunction on T_t with eigenvalue $\lambda_i(t)$ and the collection $\{\varphi_i(t) : i \in \mathbb{N}\}$ is an orthonormal basis*[12] *of $L^2(T_t)$.*

Proof. For each pair of smooth functions $f, g : T_t \to \mathbb{R}$ define

$$q_t(f,g) = \int_{T_t} \nabla f \cdot \nabla g, \quad \text{and} \quad n_t(f,g) = \int_{T_t} f \cdot g.$$

[10] Recall that a vertex is not, by our consistent definition, a critical point of u_t.

[11] By 'analytic', we mean that $t \mapsto \varphi_t \circ h_t$ is an analytic path in each Sobolev space on T_1.

[12] By 'orthonormal basis', we mean that $\int_{T_t} \varphi_i(t) \cdot \varphi_j(t) = \delta_{ij}$ and the finite linear combinations are dense in $L^2(T_t)$.

Let $\mathcal{D}_t$ be the completion of the smooth functions with respect to the norm $f \mapsto \sqrt{q_t(f,f) + n_t(f,f)}$. This space may be naturally regarded as a dense subspace of $L^2(T_t)$, and the form q_t extends to a closed form with domain $\mathcal{D}_t$. A function $u \in \mathcal{D}_t$ is an eigenfunction of the Neumann Laplacian on T_t with eigenvalue λ if and only if for each $v \in \mathcal{D}_t$ we have $q_t(u,v) = \lambda \cdot n_t(u,v)$.[13] For each labeled triangle T_t, let h_t be the unique real-affine map that sends the ordered triple $(0,1,i)$ to $(v_1(t), v_2(t), v_3(t))$. The map $f \to h_t \circ f =: h_t^*(f)$ sends smooth functions on T_t to smooth functions on T_1, and a straightforward argument shows that h_t^* is a bounded isomorphism from $L^2(T_t)$ to $L^2(T_t)$ that maps $\mathcal{D}_t$ onto $\mathcal{D}_1$. Note that for each $f, g \in L^2(T_1)$ we have $n_t(f \circ h_t^{-1}, g \circ h_t^{-1}) = 2 \cdot a_t \cdot n_1(f,g)$ where a_t is the area of T_t. For each $f, g \in \mathcal{D}_1$, define $\widetilde{q}_t(f,g) := (2a_t)^{-1} \cdot q_t(f \circ h_t^{-1}, g \circ h_t^{-1})$. By tracing through the definitions, one finds that u is a Neumann eigenfunction on T_t with eigenvalue λ iff for each $w \in \mathcal{D}_1$ we have

$$\widetilde{q}_t(u \circ h_t, w) \ = \ \lambda \cdot n_1(u \circ h_t, w).$$

The family $t \mapsto \widetilde{q}_t$ is an analytic family of type (a) in the sense of Kato (see Theorem 4.2 in Chapter VII of Kato). Moreover, the resolvent of the associated operator is compact, and hence it follows[14] that for each $i \in \mathbb{N}$, there exist an analytic path $t \mapsto \psi_i(t)$ and an analytic path $t \mapsto \lambda_i(t)$ so that for each $t \in [0,1]$, we have $\widetilde{q}_t(\psi_i(t), w) \ = \ \lambda_i(t) \cdot n_1(\psi_i(t), w)$ and the collection $\{\psi_i(t) \, : \, i \in \mathbb{N}\}$ is an orthonormal basis of $L^2(T_1)$. Set $\varphi_i(t) := (2a_t)^{-\frac{1}{2}} \cdot \psi_i(t) \circ h_t^{-1}$. □

It is important to note that the analytic eigenvalue 'branches' $t \mapsto \lambda_i(t)$ of Lemma 11.1 can not, in general, be ordered according to the size. Indeed, two eigenvalue branches λ_i and λ_j may 'cross' at some $t \in [0,1]$ in the sense that $\lambda_i(s) < \lambda_j(t)$ for $s < t$ and $\lambda_i(s) > \lambda_j(t)$ for $s > t$.

Lemma 11.2. *There exist a partition* $0 = t_0 < t_1 < \cdots < t_k = 1$ *of* $[0,1]$ *and for each* $j = 0, \ldots, k-1$, *an analytic path* $t \mapsto u_t^j$ *such that for each* $t \in [t_j, t_{j+1}]$, *the function* u_t^j *is a second Neumann eigenfunction of* T_t.

Proof. Let $s \mapsto \varphi_i(s)$ and $s \mapsto \lambda_i(s)$ be the eigenfunction and eigenvalue branches provided by Lemma 11.1. For each t there exists $i(t) \in \mathbb{N}$ so that $\mu_2(t) = \lambda_{i(t)}$. By Corollary 5.5, the dimension of the space of second Neumann eigenfunctions is at most two. Let A be the set of $t \in [0,1]$ such that $\mu_2(t)$ has multiplicity exactly equal to two. It suffices show that A is discrete. Indeed, then A would be finite, and $i(t)$ would be locally constant on the complement of A. The set A would give the desired partition.

Fix $t \in A$. Since the Neumann spectrum of T_t is discrete, there exists $\epsilon > 0$ so that $\mu_2(t)$ is the only eigenvalue of T_t that lies in $(\mu_2(t) - \epsilon, \mu_2(t) + \epsilon)$. There exist unique integers i and j so that the $\mu_2(t)$-eigenspace of Δ_t is spanned by $\varphi_i(t)$ and $\varphi_j(t)$. We have $\lambda_i(t) = \mu_2(t) = \lambda_j(t)$. By continuity of the eigenvalue branches, there exists $\delta > 0$ so that if $|s - t| < \delta$, then $\lambda_i(s)$ and $\lambda_j(s)$ are the only eigenvalues of T_s that lie in $(\mu_2(t) - \epsilon, \mu_2(t) + \epsilon)$. In particular, we have $\mu_2(s) = \min\{\lambda_i(s), \lambda_j(s)\}$ for $|s - t| < \delta$.

Define $C_t := \{s \in [0,1] : \lambda_i(s) = \lambda_j(s)\}$. Real-analyticity implies that C_t is either discrete or $C_t = [0,1]$. Thus, to finish the proof, it suffices to show that $C_t \neq [0,1]$.

Suppose to the contrary that $C = [0,1]$. Then for $|s - t| < \delta$, we have $\mu_2(s) = \lambda_i(s) = \lambda_j(s)$. Let t^* be the supremum of s so that $\mu_2(s) = \lambda_i(s) = \lambda_j(s)$. The triangle T_1 is right isosceles, the eigenvalue $\mu_2(1)$ is simple, and so $t^* < 1$. The eigenspace associated to $\mu_2(t^*)$ is two dimensional and is spanned by $\varphi_i(t^*)$ and $\varphi_j(t^*)$. Let s_k be a decreasing sequence that limits to t^*. For each k, let u_k be a $\mu_2(s_k)$-eigenfunction with L^2-norm equal to one. Note that u_k is orthogonal to the space spanned by $\varphi_i(s_k)$ and $\varphi_j(s_k)$. The sequence u_k has a subsequence that limits to a $\mu_2(t^*)$-eigenfunction u. The function u is orthogonal to the span of $\varphi_i(t^*)$ and $\varphi_j(t^*)$. But this contradicts the fact that $\varphi_i(t^*)$ and $\varphi_j(t^*)$ span the $\mu_2(t^*)$-eigenspace. □

Lemma 11.2 allows us to construct a continuous family of second Neumann eigenfunctions to which we can apply our methods. Indeed, let E_t denote the space of second Neumann eigenfunctions, and for each $j = 1, \ldots k-1$, choose a continuous path of eigenfunctions inside E_{t_j} that joins the eigenfunction $u_{t_j}^{j-1}$ to the

[13]Indeed, Neumann conditions are the 'natural boundary conditions'.

[14]See Chapter VII of [Kato], especially Remark 4.22.

eigenfunction $u^j_{t_j}$. By concatenating such paths with the paths u^j_t of Lemma 11.2 we obtain a continuous path of second Neumann eigenfunctions that joins u_0 to u_1. The methods of this paper apply to this path, and we obtain Theorem 1.1 without using simplicity.

12. Triangles with no critical point

Let u be a second Neumann eigenfunction eigenvalue $\mu_2(T)$ of a triangle T. In §10, we showed that a second Neumann eigenfunction u on a triangle has at most one critical point. By combining this with Proposition 7.1, we obtain

Theorem 12.1. *The eigenfunction u has a critical point if and only if each vertex is an isolated local extremum. Moreover, the maximum (resp. minimum) value of u is achieved only at the vertices of the triangle.*

Proof. For the second statement, note that neither the maximum nor the minimum value is achieved at a nondegenerate critical point of Morse index 1. □

Corollary 12.2. *If T is an acute triangle, then u has a critical point if and only if u does not vanish at each vertex of T. Moreover, if T is an obtuse triangle, then u has a critical point if and only if u does not vanish at the acute vertices of T and the first Bessel coefficient of u at the obtuse vertex is zero.*

Proof. This follows from Lemma 4.2 and Proposition 7.1. □

Proposition 12.3. *Let T be a nonequilateral isosceles triangle and let β be the angle of the apex of T. If $\beta > \pi/3$, then u has no critical points. If $\beta < \pi/3$, then u has exactly one critical point.*

Proof. Up to rescaling and rigid motion, each isosceles triangle may be identified with the triangle with vertices $(-t, t, i)$. If $\beta > \pi/3$, then $t > 1/\sqrt{3}$, and the second Neumann eigenvalue $\mu_2(t)$ is simple [Sdj15]. Let $t \mapsto u_t$ be the analytic family of second Neumann eigenfunctions associated to the path $t \mapsto (-t, t, i)$. Each triangle is preserved by the reflection $\sigma(x + iy) = -x + iy$ which has fixed point set $x = 0$. For each t, we have either $u_t \circ \sigma = u_t$ or $u_t \circ \sigma = -u_t$. The triangle with vertices $(-1, 1, i)$ is a right isosceles of triangle, and inspection of (21) shows that $u_1 \circ \sigma = -u_1$. Thus, by continuity, for each $t > 1/\sqrt{3}$, we have $u_t \circ \sigma = -u_t$ is anti-invariant. In particular, $u_t(i) = 0$ for each $t > 1/\sqrt{3}$, and hence by Lemma 4.2, the vertex i is not an isolated local extremum. Therefore, by Theorem 12.1, the eigenfunction u has no critical points for $t > 1/\sqrt{3}$.

For small t, the triangle may be approximated by a sector with angle $2 \cdot \arctan(t)$ and radius 1. In particular, one can show that for sufficiently small t the function satisfies $u_t \circ \sigma = u_t$ (see, for example, Proposition 2.4 in [Bnl-Brd99]). Therefore, by continuity, $u_t \circ \sigma = u_t$ for each $t < 1/\sqrt{3}$. Thus, $\partial_x u_t(0) = 0$ and 0 is a critical point of u_t for each $t < 1/\sqrt{3}$. □

Let G be the group of linear transformations of the plane generated by isometries and homotheties. If u is a second Neumann eigenfunction for a labeled triangle (v_1, v_2, v_3) and $g \in G$, then $u \circ g$ is a second Neumann eigenfunction for a labeled triangle $(g(v_1), g(v_2), g(v_3))$. In particular, $u \circ g$ has a critical point if and only if u does. Let $\mathcal{T}$ denote the quotient of the set of labeled triangles by G. That is, two labeled triangles (v_1, v_2, v_3) and (v_1', v_2', v_3') define the same point in $\mathcal{T}$ if and only if there exists $g \in G$ so that $(g(v_1), g(v_2), g(v_3)) = (v_1', v_2', v_3')$.

Let $\beta(v_i)$ be the angle at v_i. The map $(v_1, v_2, v_3) \mapsto (\beta(v_1), \beta(v_2), \beta(v_3))$ defines a bijection onto the interior of the convex hull of $(\pi, 0, 0), (0, \pi, 0)$ and $(0, 0, \pi)$ in $\mathbb{R}^3$. See Figure 2.[15] We equip $\mathcal{T}$ with the topology and real-analytic structure of this simplex.

[15]See https://polymathprojects.org/tag/polymath7/ for the same picture together with a description of the triangles that were known to have no hot spots as of 2013.

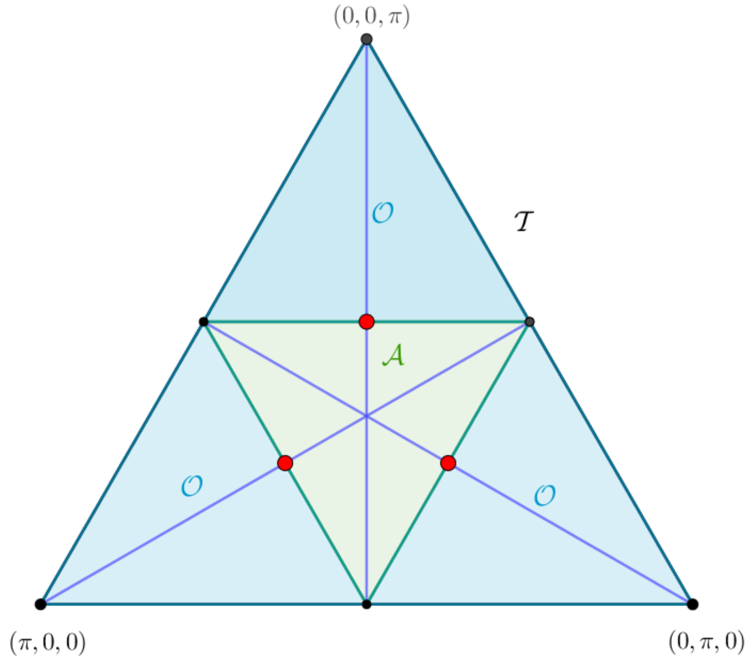

FIGURE 2. The 'Teichmüller space' of labeled triangles. The three red points correspond to right isosceles triangles. The light blue regions correspond to the set $\mathcal{O}$ of obtuse triangles. The light green region corresponds to the set $\mathcal{A}$ of acute triangles.

Let $\mathcal{A}$ and $\mathcal{O}$ respectively denote the subspace of $\mathcal{T}$ consisting of (equivalence classes of) acute and obtuse triangles. Let $\mathcal{C}$ denote set of (equivalence classes of) triangles T such that each eigenfunction corresponding to the first Neumann eigenvalue of T has a critical point.[16]

Theorem 12.4. *The set $\mathcal{C}$ is open in $\mathcal{T}$, the set $\mathcal{C} \cap \mathcal{A}$ is dense in $\mathcal{A}$, and $\mathcal{C} \cap \mathcal{O}$ is empty.*

Proof. Suppose that $T \in \mathcal{C}$. Let u be a second Neumann eigenfunction of T. By Theorem 1.1, the function u has exactly one critical point, p, and, by Corollary 6.6, the point p belongs to a side e of T and is nondegenerate. Let $\widetilde{u}$ be the extension of u to the kite K_e. Since p is nondegenerate, by Lemma 9.1 the critical point is stable under a small perturbation of T. In particular, $\mathcal{C}$ is an open subset of $\mathcal{T}$.

Let T^* denote the equilateral triangle. For each $T \neq T^*$, the vector space E_T of second Neumann eigenfunctions of T is one dimensional subspace of $L^2(T^*)$ [Sdj15] [Mym13] [Atr-Brd04]. In particular, we have a real line bundle $\mathcal{E}$ over the punctured simplex $\mathcal{T} - \{T^*\}$ such that the fiber over T equals E_T. Let $\mathcal{S} \to \mathcal{T}$ denote the associated 'sphere bundle'. That is, the fiber of $\mathcal{S}$ over T consists of the two eigenfunctions in E_T whose $L^2(T^*)$-norm equals one.

Let $\mathcal{U}$ be the subset of $\mathcal{T} - \{T^*\}$ obtained by removing the segment $\mathcal{L}$ that joins $T^* = (\pi/3, \pi/3, \pi/3)$ to $(\pi/2, \pi/2, 0)$. The set $\mathcal{U}$ is simply connected, and hence the bundle $\mathcal{S}$ is trivial over $\mathcal{U}$. In particular, there are exactly two sections of $\mathcal{S}$ defined over $\mathcal{U}$. Let $T \to u(T)$ denote one of the sections defined over $\mathcal{U}$. The angles (β_1, β_2) of T at the labeled vertices v_1 and v_2 provide coordinates for $\mathcal{T}$. For each fixed β_2, standard perturbation theory implies that the map $\beta_1 \mapsto u(\beta_1, \beta_2)$ is analytic (away from $\mathcal{L}$), and similarly for each fixed β_1 the map $\beta_2 \mapsto u(\beta_1, \beta_2)$ is analytic. Therefore, Hartog's separate analyticity theorem implies that $T \to u(T)$ is analytic on $\mathcal{U}$.

In particular, for each i, the value of $u(T)$ at the vertex v_i is a real-analytic function on $\mathcal{U}$. By using (21), we find that $u(\pi/2, \pi/4, \pi/4)(v_2) \neq 0$, $u(\pi/4, \pi/2, \pi/4)(v_3) \neq 0$, and $u(\pi/4, \pi/4, \pi/2)(v_1) \neq 0$. Therefore, for

[16] Since $\mu_2(T)$ is simple unless T is the equilateral triangle [Sdj15], the set $\mathcal{C} \cap \mathcal{A}$ equals the set of triangles such that at least one second Neumann eigenfunction has a critical point.

each i, the map $T \mapsto u(T)(v_i)$ is nonzero on a dense subset of $\mathcal{U}$. Corollary 12.2 then implies that the set $\mathcal{C} \cap \mathcal{A}$ is dense in $\mathcal{A}$.

By Corollary 12.2, the set $\mathcal{O} \cap \mathcal{C} \cap \mathcal{U}$ is contained in the set of T such that the first Bessel coefficient $c_1(T)$ of $u(T)$ is zero at the obtuse vertex. The map $T \mapsto c_1(T)$ is a real-analytic function on $\mathcal{U}$. By Proposition 12.3, if T is an obtuse isosceles triangle, then $u(T)$ has no critical point, and hence $c_1(T) \neq 0$. Thus c_1 is nonzero on an open dense subset of $\mathcal{U}$ which, in turn, is open and dense in $\mathcal{T}$. Hence $\mathcal{O} \cap \mathcal{C}$ is nowhere dense. But from above we have that $\mathcal{C}$ is open and hence $\mathcal{O} \cap \mathcal{C}$ is open. Therefore $\mathcal{O} \cap \mathcal{C}$ is empty. ☐

Corollary 12.5. *If T is a right triangle, then u does not have a critical point.*

We end the article with the following conjecture.

Conjecture 12.6. *If T is not an equilateral triangle, then a second Neumann eigenfunction of T has a critical point if and only if T is an acute triangle that is not isosceles with apex angle greater than $\pi/3$.*

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