

RIGIDITY OF THE LENGTH-ANGLE SPECTRUM FOR CLOSED HYPERBOLIC SURFACES

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ABSTRACT. We introduce a new *unmarked* geometric spectrum, the *length-angle spectrum*, for closed hyperbolic surfaces. We show that unlike the *length spectrum*, the length-angle spectrum of any closed hyperbolic surface determines the surface, up to isometry.

1. INTRODUCTION

Let S_g be a closed orientable Riemannian surface of genus g . For $g \geq 2$ there are many hyperbolic metrics that S_g can be equipped with. Due to this sharp contrast to Mostow's rigidity theorem in higher dimensions it is an important question what kind of geometric information of a hyperbolic metric on S_g can uniquely determine the metric, up to isometry.

One set of important geometric collection to look at is the set of unit speed closed geodesics. From here onwards by geodesic we always mean unit speed geodesic. Lengths of these closed geodesics with an explicit marking, called the *marked length spectrum* (see §1.1, [10], [18], [8]), is known to determine the metric, up to isometry. Without the marking such a result is known to be not true in general (see §1.1, [21], [20]). We introduce the length-angle spectrum for these surfaces in §1.4. This spectrum, in a weak sense, keeps track of how pairs of simple closed geodesics intersect with each other. As in the case of length spectrum, this spectrum do not come with an explicit marking. However, in our main result of this paper we show that the length-angle spectrum determines the metric, up to isometry.

Main Theorem. *Let S, S' be two closed hyperbolic surfaces with identical length-angle spectrum. Then S is isometric to S' .*

1.1. Length spectrum. Let $\mathcal{C}(S_g)$ be the space of closed curves on S_g up to homotopy. Let \mathcal{M}_g denote the *moduli space* of hyperbolic metrics on S_g up to isometry. For $S \in \mathcal{M}_g$ and $\gamma \in \mathcal{C}(S_g)$ let $\ell_S(\gamma)$ denote the length of the closed geodesic freely homotopic to γ on S . Since every closed essential curve on S is freely homotopic to a unique closed geodesic, we denote by $\mathcal{C}(S)$ the space of all closed geodesics on S . The marked length spectrum of S is defined to be the map

$$\ell : \mathcal{C}(S) \rightarrow \mathbb{R}$$

which sends a geodesic in $\mathcal{C}(S)$ to its length. A classical result dating back to Fricke-Klein [10] says that the marked length spectrum of a hyperbolic metric on S_g determines the metric. In [3] the possibility of extensions of this rigidity result to negatively curved manifolds was first suspected. It was subsequently confirmed

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by Otal [18] and Croke [8] for surfaces with variable negative curvature. Since then it has been extended in many different directions.

The sequence of lengths of all closed geodesics on S , counting multiplicity and without any marking, is called the *length spectrum* of the surface. There is a well-known connection between the length and the Laplace spectrum of S due to Huber [13]: *The length and the Laplace spectrum of a hyperbolic metric on S_g determines each other.* Rigidity results related to the length spectrum probably appeared for the first time in the work [11] of I. M. Gel'fand who showed that there is no continuous family in \mathcal{M}_g for which the length spectra stay the same. He further conjectured that the length spectrum determines the hyperbolic metric. The first counter examples to this conjecture appear in [21]. Later Sunada [20] gave an elegant method for constructing such counter examples.

Remark 1.1. *Rigidity questions related to (unmarked) length spectrum can be traced back to the question popularized by M. Kac [15]: ‘Can one hear the shape of a drum?’*

The best possible result for the length spectrum is due to McKean [17] that says that only finitely many non-isometric closed hyperbolic surfaces of a fixed genus can have the same length spectrum i.e. the map from $L : \mathcal{M}_g \rightarrow \mathbb{R}^{\mathbb{N}}$ that sends a metric to its length spectrum is ‘finite to one’. In [22], Wolpert gave another proof of this fact. He moreover showed that there is a proper analytic sub-variety \mathcal{V}_g of \mathcal{M}_g that contains all genus g hyperbolic surfaces that are not determined by their length spectrum. Hence Gel'fand's conjecture, although false in general, is true in the generic sense.

Remarks 1.2. 1. *For a closed surface S equipped with a negatively curved metric it is long known that the geodesic flow on the unit tangent bundle of S is ergodic [1]. This fact implies that closed geodesics are dense on S . This, in some sense, was the reason for the above rigidity speculations and results.*

2. *On the contrary, simple closed geodesics on a closed hyperbolic surface is nowhere dense [2]. Hence there is no reason, along these lines, to believe that Theorem 1 should be true. However, since the length-angle spectrum is ‘too’ large a collection compared to the length spectrum it is probably not that surprising that Theorem 1 is true.*

3. *We shall see that the proof of Theorem 1 follows a different approach from the approaches in [18], [8] and is somewhat motivated by the classical arguments in the proof of the $9g - 9$ Theorem [9].*

1.2. Simple length spectrum. Let $\mathcal{G}(S_g) \subset \mathcal{C}(S_g)$ be the space of simple closed curves on S_g up to homotopy. $\mathcal{G}(S_g)$ has been an important object of study in the literature (see [9], [16]). For $S \in \mathcal{M}_g$ let $\mathcal{G}(S) \subset \mathcal{C}(S)$ be the space of simple closed geodesics on S . The sequence of lengths of these simple closed geodesics on S , counting multiplicity, is called the *simple length spectrum* of S .

Wolpert's proof in [22] of McKean's result [17] applies to the simple length spectrum providing that for a fixed g there are only finitely many surfaces in \mathcal{M}_g that have the same simple length spectrum. It would be very interesting to see how far can the simple length spectrum determine the surface. Since the whole length spectrum is not sufficient to determine the metric, it is probably true that simple length spectrum is not sufficient either (see §7.2). The author could not locate any literature involving this question.

1.3. Angles between simple closed geodesics. Now we consider another geometric *collection* related to pairs of simple closed geodesics, the angles between them. Suppose $\gamma, \delta \in \mathcal{G}(S)$ intersect each other $\iota(\gamma, \delta)$ times. We consider the set

$\Theta_S(\gamma, \delta)$ of $\iota(\gamma, \delta)$ angles of intersection between γ and δ where each angle is measured in the counter-clockwise direction from γ to δ and the sequence is recorded along γ as they occur.

Defined that way $\Theta_S(\gamma, \delta)$ is a point in the product of $\iota(\gamma, \delta)$ copies of $(0, \pi)$. Since we do not record the points of intersection between γ and δ it is clear that $\Theta_S(\gamma, \delta)$ is defined only up to the action of the cyclic permutation $(1, 2, \dots, \iota(\gamma, \delta))$ and its inverse. The set of angles in $\Theta_S(\gamma, \delta)$ forgetting the ordering and multiplicities would be denoted by $\Phi(\gamma, \delta)$.

Remark 1.3. For a collection of simple closed geodesics and simple geodesic arcs $\alpha, \beta_1, \dots, \beta_k$ on S the above definition equally works to define $\Theta_S(\alpha, \cup_{i=1}^k \beta_i)$. When α is a geodesic arc we list the angles beginning from one of the end points of α through the other.

Definition 1.3.1. A tuple $(\theta_1, \dots, \theta_l)$ is called an ordered subset of the angle set $\Theta_S(\alpha, \cup_{i=1}^k \beta_i)$ if the ordering of θ_i s respect that of $\Theta_S(\alpha, \cup_{i=1}^k \beta_i)$.

Angles between more general types of geodesics has been studied in the literature. One particular case is the self-intersection angles of non-simple closed geodesics. An interesting statistical behavior of these self-intersection angles was obtained by Pollicott and Sharp in [19].

1.4. Length-Angle Spectrum. The main object of our study in this paper is the length-angle spectrum $\mathcal{L}\Theta(S)$ that we define as the collection

$$\{(\ell_S(\gamma), \ell_S(\delta), \Theta_S(\gamma, \delta)) : \gamma, \delta \in \mathcal{G}(S)\}.$$

In a weak sense we do have a marking in $(\ell_S(\gamma), \ell_S(\delta), \Theta_S(\gamma, \delta))$ when γ, δ are known. When we consider $\mathcal{L}\Theta(S)$ as a whole, we do not have any explicit idea which pair of geodesics correspond to which tuple in $\mathcal{L}\Theta(S)$ in the sense of marked length spectrum.

1.5. Sketch of the proof. We now give a brief idea of the proof. Consider two simple closed geodesics α and β on S with $\iota(\alpha, \beta) = 1$. It is not difficult to see that a thickened neighborhood of $\alpha \cup \beta$ in S determines a unique compact one holed torus $\mathcal{T}(\alpha, \beta) \subset S$ with geodesic boundary. A simple but important observation is that the triple $(\ell_S(\beta_1), \ell_S(\beta_2), \Theta_S(\alpha, \beta))$ determines $\mathcal{T}(\alpha, \beta)$. In particular if α' and β' be two simple closed geodesics on another hyperbolic surface S' with

$$(\ell_S(\alpha), \ell_S(\beta), \Theta_S(\alpha, \beta)) = (\ell_{S'}(\alpha'), \ell_{S'}(\beta'), \Theta_{S'}(\alpha', \beta'))$$

then the corresponding compact one holed torus with geodesic boundary $\mathcal{T}(\alpha', \beta')$ in S' is an isometric copy of $\mathcal{T}(\alpha, \beta)$ (see Lemma 3.0.1). This observation works as the motivation for our approach. In the first step towards the proof of Theorem 1 we formulate a rigidity criteria of similar type that works for the whole surface. We consider a simple closed non-separating geodesic γ_0 on S and construct a pants decomposition of S such that different geodesics in the pants decomposition are distinguishable from the angles they make with γ_0 . More precisely,

Theorem 1.3.1. *There is a pants decomposition $\mathcal{P}_0 = \{\alpha_i : i = 1, 2, \dots, 3g - 3\}$ of S that satisfies:*

- (1) γ_0 and α_i intersect minimally i.e. for α_i non-separating $\iota(\gamma_0, \alpha_i) = 1$ and for α_i separating $\iota(\gamma_0, \alpha_i) = 2$, and
- (2) the $\Phi(\gamma_0, \alpha_i)$ s are mutually disjoint i.e. $\Phi(\gamma_0, \alpha_i) \cap \Phi(\gamma_0, \alpha_j) = \emptyset$ for $i \neq j$.

Remark 1.4. *We shall see in the proof of this theorem that there are infinitely many such pants decompositions. It would be interesting to see if one can construct a pants decomposition that makes distinct angles at each intersection. In our construction, for α_i separating, the two angles in $\Phi(\gamma_0, \alpha_i)$ may be identical.*

With such a pants decomposition at hand, our marked rigidity result is:

Theorem 1.4.1. *Let S' be a closed hyperbolic surface of genus g . Let γ'_0 be a simple closed geodesic on S' and $\mathcal{P}'_0 = \{\alpha'_i : i = 1, \dots, 3g-3\}$ be a pants decomposition of S' such that for each $i = 1, 2, \dots, 3g-3$:*

- (1) $\ell_S(\alpha_i) = \ell_{S'}(\alpha'_i)$,
- (2) $\Theta_S(\gamma_0, \alpha_i) = \Theta_{S'}(\gamma'_0, \alpha'_i)$ and
- (3) $\Theta_S(\gamma_0, \cup_{i=1}^{3g-3} \alpha_i) = \Theta_{S'}(\gamma'_0, \cup_{i=1}^{3g-3} \alpha'_i)$

where γ_0 and $\mathcal{P}_0 = \{\alpha_i : i = 1, 2, \dots, 3g-3\}$ are as in Theorem 1.3.1. Then S' is isometric to S .

The idea now is to find a way of extracting information about γ_0 and \mathcal{P}_0 from the length-angle spectrum $\mathcal{L}\Theta(S)$. For that we consider a sequence of simple closed geodesics $T_{\bar{v}_n}(\gamma_0)$ indexed by $\bar{v}_n = (v_{1,n}, \dots, v_{3g-3,n}) \in \mathbb{Z}_+^{3g-3}$. This $T_{\bar{v}_n}(\gamma_0)$ is the geodesic freely homotopic to the simple closed curve

$$D_{\alpha_1}^{v_{1,n}} \circ D_{\alpha_2}^{v_{2,n}} \circ \dots \circ D_{\alpha_{3g-3}}^{v_{3g-3,n}}(\gamma_0),$$

that is obtained from γ_0 by applying $D_{\alpha_i}^{v_{i,n}}$, $v_{i,n}$ Dehn twists (see §2) along α_i , to γ_0 for $i = 1, 2, \dots, 3g-3$. Here the α_i s are as in Theorem 1.3.1. In §3 we show that the corresponding angle sets $\Theta_S(\gamma_0, T_{\bar{v}_n}(\gamma_0))$ in a specific manner encodes a lot of information that we need about the α_i s. To give an idea of the type of information these angle sets encode we begin by the following.

Definition 1.4.1. *For any finite set A we denote the cardinality of A by $\#|A|$. Two diverging sequences of integers u_n and v_n are called similar, denoted $u_n \approx v_n$, if there is a $k \in \mathbb{N}$ such that $|u_n - v_n| \leq k$.*

Let α, β be two simple closed geodesics on S with $\iota(\alpha, \beta) = 1$. Let $\beta_n = T_{\alpha}^n(\beta)$, the geodesic freely homotopic to $D_{\alpha}^n(\beta)$, and let $\mathcal{C}_{\alpha} \subset S$ denote the collar neighborhood around α .

Theorem 1.4.2. *Let $\Theta_S(\alpha, \beta) = (\phi)$ and $\Theta_S(\beta|_{\mathcal{C}_{\alpha}}, \beta_n|_{\mathcal{C}_{\alpha}}) = (\theta_1^n, \theta_2^n, \dots, \theta_m^n)$. Then:*

- (1) *there is a partial monotonicity among θ_j^n :*

- (a) *if $n > 0$ then*

$$\theta_1^n > \theta_2^n > \dots > \phi < \dots < \theta_{m-1}^n < \theta_m^n$$

- (b) *if $n < 0$ then*

$$\theta_1^n < \theta_2^n < \dots < \phi > \dots > \theta_{m-1}^n > \theta_m^n$$

where the angles before and after ϕ correspond to the intersections between $\beta|_{\mathcal{C}_{\alpha}}$ and $\beta_n|_{\mathcal{C}_{\alpha}}$ in the two different halves of $\mathcal{C}_{\alpha} \setminus \alpha$,

- (2) *for any $\epsilon > 0$: $\#\{i : \theta_i^n \in (\phi - \epsilon, \phi + \epsilon)\} \approx n$.*

Remark 1.5. *We shall see in §4 that the numbers of θ_i^n s that appear before and after ϕ , in part (1) of the theorem, differ by at most two.*

Given the above, the complete arguments of the proof go as follows. Let S' be another closed hyperbolic surface of genus g' with $\mathcal{L}\Theta(S) = \mathcal{L}\Theta(S')$. Without loss of generality we may assume that $g \geq g'$. Since $\mathcal{L}\Theta(S) = \mathcal{L}\Theta(S')$, for each n , there are simple closed geodesics γ'_n, δ'_n on S' such that

$$(\ell_S(\gamma_0), \ell_S(T_{\bar{v}_n}(\gamma_0)), \Theta_S(\gamma_0, T_{\bar{v}_n}(\gamma_0))) = (\ell_{S'}(\gamma'_n), \ell_{S'}(\delta'_n), \Theta_{S'}(\gamma'_n, \delta'_n)).$$

A priori the sequence γ'_n depends on n . Since the number of simple closed geodesics on S' of length $\ell_S(\gamma_0)$ is finite, up to extracting a subsequence, we have a fixed closed geodesic $\gamma'_0 \in \mathcal{G}(S')$ such that

$$(\ell_S(\gamma_0), \ell_S(T_{\bar{v}_n}(\gamma_0)), \Theta_S(\gamma_0, T_{\bar{v}_n}(\gamma_0))) = (\ell_{S'}(\gamma'_0), \ell_{S'}(\delta'_n), \Theta_{S'}(\gamma'_0, \delta'_n)).$$

The last part of the proof studies the sequence δ'_n . Up to extracting a subsequence, these geodesics converge to a *geodesic lamination* L . One of the most important step, for the rest of the proof, is to use the structure of $\Theta_{S'}(\gamma'_0, \delta'_n) = \Theta_S(\gamma_0, T_{\bar{v}_n}(\gamma_0))$ from §4 to deduce that each *minimal sub-lamination* of L is a simple close geodesic. With the assumption

$$\lim_{n \rightarrow \infty} \frac{v_{i+1,n}}{v_{i,n}} = 0, \text{ for each } i = 1, 2, \dots, 3g - 2$$

we show, using results from §4, that each half-leaf of L spirals around some simple closed geodesic α'_i and the collection (α'_i) contains at least $3g - 3$ simple closed geodesics. Since α'_i s are mutually non-intersecting and $g' \leq g$ it follows that, in fact $g' = g$ and the collection $\{\alpha'_1, \dots, \alpha'_{3g-3}\}$ forms a pants decomposition \mathcal{P}'_0 of S' . Further analysis of the sequence δ'_n via the convergence $\delta'_n \rightarrow L$ provides us the following theorem that concludes the proof of Theorem 1 via Theorem 1.4.1.

Theorem 1.5.1. *Let γ'_0 and $\mathcal{P}'_0 = \{\alpha'_i : i = 1, \dots, 3g - 3\}$ respectively be the simple closed geodesic on S' and the pants decomposition of S' as above. Then for each $i = 1, 2, \dots, 3g - 3$:*

- (1) $\ell_{S'}(\alpha'_i) = \ell_S(\alpha_i)$,
- (2) $\Theta_{S'}(\gamma'_0, \alpha'_i) = \Theta_S(\gamma_0, \alpha_i)$ and
- (3) $\Theta_{S'}(\gamma'_0, \cup_{i=1}^{3g-3} \alpha'_i) = \Theta_S(\gamma_0, \cup_{i=1}^{3g-3} \alpha_i)$,

where γ_0 and $\mathcal{P}_0 = \{\alpha_i : i = 1, 2, \dots, 3g - 3\}$ are as in Theorem 1.3.1.

1.6. Structure of the article. In §2 we recall some basic concepts and tools that we are going to use in the later sections. Among other things, we recall (i) formal definition of Dehn twist and (ii) the structure theorem for geodesic laminations. The next section is devoted to two rigidity results Lemma 3.0.1 and Theorem 1.3.1. We give proofs of these two results there. The next section §4 is the most important section of this article from the technical point of view. We begin this section by recalling asymptotic growth of the intersection numbers $\iota(\gamma, T_{n_1, \dots, n_k}(\beta))$ for simple closed geodesics $\alpha_1, \dots, \alpha_k, \beta, \gamma$ with α_i s mutually disjoint. We then use this asymptotic to study asymptotic growth of lengths of geodesics of the form $\ell(T_{n_1, \dots, n_k}(\beta))$. Later in that section we develop qualitative and asymptotic properties of angle sets $\Theta_S(\gamma, T_{n_1, \dots, n_k}(\beta))$. The next section, §5, is devoted to the construction of the pants decomposition in Theorem 1.3.1. We prove Theorem 1 in §6. In the end we have an appendix where we explain some results that are used in various parts of the article that are most likely known to experts.

2. PRELIMINARIES

In this section we review some standard facts from the geometry of hyperbolic surfaces that will be used in our study. The area formula of a hyperbolic geodesic polygon is the simplest among these. Let G be a hyperbolic geodesic n -gon with interior angles ϕ_1, \dots, ϕ_n . Then the area $|G|$ of G is given by

$$|G| = (n - 2)\pi - \sum_{i=1}^n \phi_i. \quad (2.1)$$

2.1. Collars. Let α be a simple closed geodesic on S . The Collar Theorem says that there is a *collar* neighborhood $\mathcal{C}_\alpha \subset S$ of α which is isometric to the cylinder $[-w(\alpha), w(\alpha)] \times \mathbb{S}^1$ with the metric $dr^2 + \ell_\alpha^2 \cosh^2 rd\theta^2$ and for any two non-intersecting simple closed geodesics α, β the collars $\mathcal{C}_\alpha, \mathcal{C}_\beta$ are mutually disjoint. The coordinates (r, θ) on \mathcal{C}_α via this isometry are called the *Fermi coordinates*. For an $x \in \mathcal{C}_\alpha$ let $(r(x), \theta(x))$ be its Fermi coordinates. Then $r(x)$ denotes the signed

distance of x from α and $\theta(x)$ denotes the projection of x on α when α is identified with \mathbb{S}^1 [4, p-94].

2.2. Dehn twist. Dehn twist homeomorphisms are the most important tools used in this paper. We use them, for example, to construct our sequence of simple closed geodesics $T_{\bar{v}_n}(\gamma_0)$. For a more complete and detailed discussion of these we refer the readers to [9, Chapter 3].

Let α be a simple closed geodesic on S and $\mathcal{C}_\alpha \subseteq S$ be the collar neighborhood of α . Fix a set of Fermi coordinates to identify \mathcal{C}_α with $[-w(\alpha), w(\alpha)] \times \mathbb{S}^1$. Take $T \leq w(\alpha)$ and consider the homeomorphism D_α of $[-T, T] \times \mathbb{S}^1 \subseteq \mathcal{C}_\alpha$ given by

$$(r, \theta) \rightarrow (r, \theta + \pi - \frac{\pi}{T} \cdot r).$$

Since D_α fixes the two boundary circles $\{-T\} \times \mathbb{S}^1$ and $\{T\} \times \mathbb{S}^1$ pointwise, it can be extended to the rest of the surface as identity. This homeomorphism (up to isotopy) is called the Dehn twist around α . For a simple closed geodesic β by $\mathcal{D}_\alpha(\beta)$, called the Dehn twist of β along α , we mean the simple closed geodesic freely homotopic to $D_\alpha(\beta)$. It is a standard fact that $\mathcal{D}_\alpha(\beta) \neq \beta$ iff $\iota(\alpha, \beta) \neq 0$ [9, Proposition 3.2].

2.3. End-to-end geodesic arcs. Fix a set of Fermi coordinates (r, θ) on \mathcal{C}_α to identify it with $[-w(\alpha), w(\alpha)] \times \mathbb{S}^1$. With respect to this identification we consider the curves in \mathcal{C}_α that are graphs of smooth maps $[-w(\alpha), w(\alpha)] \rightarrow \mathbb{S}^1$. We call them *end-to-end arcs*. When such a curve is a geodesic we call it an *end-to-end geodesic arc*. So, an end-to-end geodesic arc is basically a simple geodesic arc in \mathcal{C}_α that joins two points one in each component of $\partial\mathcal{C}_\alpha$. The end-to-end geodesic arc with constant θ coordinate equal to ϕ is called the ϕ -*radial arc* and is denoted by η_ϕ . An end-to-end geodesic arc that does not intersect at least one radial arc is called an *almost radial arc*. For an end-to-end geodesic arc ξ by $\mathcal{D}_\alpha(\xi)$ we denote the end-to-end geodesic arc that is homotopic to $D_\alpha(\xi)$ under the end point fixing homotopy.

Remark 2.2. *For any end-to-end arc γ in \mathcal{C}_α there is a unique end-to-end geodesic arc χ in \mathcal{C}_α that is homotopic to γ under the end point fixing homotopy. Also, the number of intersection $\iota(\gamma_1, \gamma_2)$ between any two end-to-end arcs γ_1, γ_2 in \mathcal{C}_α is at least the number of intersection $\iota(\chi_1, \chi_2)$ between their respective geodesic representatives χ_1, χ_2 under the end point fixing homotopy. These two facts can be seen easily by taking lifts of the involved curves to \mathbb{H}^2 .*

Remark 2.3. *Let $\partial_1^\alpha = \{-w(\alpha)\} \times \mathbb{S}^1$ and $\partial_2^\alpha = \{w(\alpha)\} \times \mathbb{S}^1$ be the two components of $\partial\mathcal{C}_\alpha$. Let $s_1 = (-w(\alpha), \theta_1) \in \partial_1^\alpha$ and $s_2 = (w(\alpha), \theta_2) \in \partial_2^\alpha$. It is not difficult to observe that:*

- (1) *if the $\theta_1 = \theta_2 = \phi$ then there is exactly one almost radial arc, η_ϕ , with end points $s_i \in \partial_i^\alpha$ and*
- (2) *if the $\theta_1 \neq \theta_2$ then there are exactly two simple paths in \mathbb{S}^1 that join θ_1 and θ_2 each of which produce exactly one almost radial arc with end points $s_i \in \partial_i^\alpha$. One of these two arcs is a Dehn twist of the other along α .*

2.3.1. Orientation of an end-to-end arc. Observe that the end-to-end arc $D_\alpha(\eta_\phi)$ is the graph of the map $\Psi : [-w(\alpha), w(\alpha)] \rightarrow \mathbb{S}^1$ that sends t to $(\phi + \pi - \frac{\pi}{T} \cdot t)$. We consider the orientation of \mathbb{S}^1 such that this map is orientation preserving. Observe that this orientation does not depend on ϕ but depends on the Fermi coordinates.

Fix a set of Fermi coordinates on \mathcal{C}_α . For an end-to-end arc ξ in \mathcal{C}_α we consider the smooth function $\Psi_\xi : [-w(\alpha), w(\alpha)] \rightarrow \mathbb{S}^1$ whose graph is ξ . We say that the orientation of ξ is positive (or negative) if Ψ_ξ is orientation preserving (or orientation reversing).

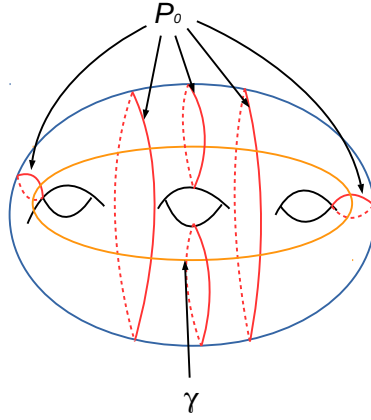


FIGURE 1. Pants decomposition

2.4. Pants decomposition. For us a pants decomposition of a hyperbolic surface S is a collection of mutually disjoint simple closed geodesics that divide the surface into three holed spheres. In §2 and thereafter we shall consider pants decompositions of closed hyperbolic surfaces that intersect a given simple closed non-separating geodesic minimally. Figure 1 is an example of such a hyperbolic surface M of genus 3. The simple closed geodesic γ is given by the yellow curve and the pants decomposition P is given by the red curves.

2.5. Geodesic laminations. A geodesic lamination on a hyperbolic surface S is a closed set \mathcal{L} which is a disjoint union of complete geodesics, called *leaves* of the lamination. The simplest examples of such things are simple closed geodesics. A little more complicated and most used one in this article are the *limits* of simple closed geodesics under repeated Dehn twists. For a more complete and detailed discussion of this topic we refer the readers to [7]. One of the most important use of these laminations come with an associated measure, and the pair is called a *measured geodesic lamination*. We shall not use the latter in this article.

2.5.1. A topology on $\mathcal{GL}(S)$. Let $\mathcal{GL}(S)$ denote the space of all geodesic laminations on S . The Chabauty topology on $\mathcal{GL}(S)$ is the restriction of the Chabauty topology on the space $\mathcal{C}(S)$ of all closed subsets of S . For detailed discussion of this topology we refer to [7, Chapter I.3.1, Chapter I.4].

Remark 2.4. *It is well-known that $\mathcal{GL}(S)$ is compact, separable and metrizable with respect to the Chabauty topology.*

Definition 2.4.1. (1) *A subset of a geodesic lamination which itself is a geodesic lamination is called a sub-lamination. A sub-lamination is called proper if it is not the whole lamination.*

(2) *A lamination is called minimal if it does not have any proper sub-lamination. A minimal sub-lamination of a lamination is a sub-lamination which is minimal as a lamination.*

(3) *A leaf of a lamination is called isolated if each point on it has a neighborhood (in S) that do not intersect any other leaf of the lamination.*

- (4) We say that a leaf l spirals along one of its ends around a lamination K if every lift of l to \mathbb{H}^2 shares an end point (at $\partial\mathbb{H}^2$) with an end point of one of the leaves of some lift of K to \mathbb{H}^2 .

In §5 we would have to deal with geodesic laminations without having prior knowledge of their structure. In that situation we shall need the following structure theorem of geodesic laminations [7, I.4.2.8. Theorem: Structure of lamination] for further understanding our geodesic laminations.

Theorem 2.4.1. *Let \mathcal{L} be a geodesic lamination on a hyperbolic surface S of finite area. Then \mathcal{L} consists of disjoint union of finitely many minimal sub-laminations K_1, \dots, K_m and finitely many isolated leaves l_1, \dots, l_p such that each l_i along one of its ends spirals around one of the K_j .*

For a simple closed geodesic (or a geodesic arc) γ and a geodesic lamination L we can consider the angle set $\Theta_S(\gamma, L)$ and the set of angles $\Phi(\gamma, L)$ in $\Theta_S(\gamma, L)$ in exactly the same way as before. In this case however $\Theta_S(\gamma, L) \in [0, \pi]^\omega$ where ω is the cardinality of $\iota(\gamma, L)$. When the last cardinality is infinite we have accumulation points in $\Phi(\gamma, L)$.

Definition 2.4.2. *We call a point $\phi \in \Phi(\gamma, L)$ a accumulation point of $\Theta_S(\gamma, L)$ if there is an ordered sequence (θ_n) in $\Theta_S(\gamma, L)$ that converges to ϕ . An ordered set (ϕ_1, \dots, ϕ_k) of $\Theta_S(\gamma, L)$ is called the set of accumulation points of $\Theta_S(\gamma, L)$ if ϕ_1, \dots, ϕ_k represent all accumulation points of $\Theta_S(\gamma, L)$ counting multiplicity.*

2.5.2. *Spiraling around a collection of geodesics.* Let $\alpha_1, \dots, \alpha_k$ be a collection of mutually disjoint simple closed geodesics. Let γ_0 be a simple closed geodesic such that $\iota(\alpha_i, \gamma_0) \neq 0$ for each $i = 1, \dots, k$. For $(n_1, \dots, n_i) \in \mathbb{Z}^k$ consider the geodesic $T_{n_1, i, \dots, n_k, i}(\gamma_0)$ that is freely homotopic to

$$D_{\alpha_1}^{n_1} \circ D_{\alpha_2}^{n_2} \circ \dots \circ D_{\alpha_k}^{n_k}(\gamma_0).$$

Now let $(n_{1,i}, \dots, n_{k,i}) \in \mathbb{Z}^k$ be a sequence such that the sign of $n_{j,i}$ be independent of i . Denote the sign of $n_{j,i}$ by $\text{sign}(j)$. As $n_{j,i}$ tend to (positive or negative) infinity $T_{n_{1,i}, \dots, n_{k,i}}(\gamma_0)$ converges to a geodesic lamination

$$L_{\text{sign}(1)\alpha_1, \dots, \text{sign}(k)\alpha_k}(\gamma_0)$$

that has exactly k closed leaves $\alpha_1, \dots, \alpha_k$. Rest of the leaves of this lamination are isolated and so along each of their ends they spiral around one of the α_i . Let ℓ be one such half-leaf that spirals around α_i . In §1.3.1 we defined the orientation of an end-to-end geodesic arc given fixed Fermi coordinates on \mathcal{C}_{α_i} . In a very similar way we can define the orientation for ℓ . Observe that as ℓ spirals around α_i the orientation of ℓ is positive if $\text{sign}(i)$ is positive and negative if $\text{sign}(i)$ is negative.

3. MARKED RIGIDITY

In this section we prove the two rigidity results Lemma 3.0.1 and Theorem 1.4.1. These results are motivated by the marked length rigidity of closed hyperbolic surfaces due to Fricke-Klein [10]. As warm up exercise we treat compact hyperbolic surfaces that are once holed tori with geodesic boundary. It is long known [6] (see also [12]) that the length spectrum of any such surface determines the surface, up to isometry.

Lemma 3.0.1. *Let \mathcal{T} and \mathcal{T}' be two compact one holed torus with geodesic boundary. Let (α, β) and (α', β') be two pairs of simple closed geodesics on \mathcal{T} and \mathcal{T}' respectively with $\iota(\alpha, \beta) = 1 = \iota(\alpha', \beta')$. If*

$$(\ell_{\mathcal{T}}(\alpha), \ell_{\mathcal{T}}(\beta), \Theta_{\mathcal{T}}(\alpha, \beta)) = (\ell_{\mathcal{T}'}(\alpha'), \ell_{\mathcal{T}'}(\beta'), \Theta_{\mathcal{T}'}(\alpha', \beta'))$$

then \mathcal{T} is isometric to \mathcal{T}' .

Proof. We use cut and paste method to prove the lemma. We begin by cutting \mathcal{T} along α . This would result in a Y -piece $\mathcal{Y}(\ell_{\mathcal{T}}(\alpha), \ell_{\mathcal{T}}(\alpha), \ell_0)$ where the triple marks the lengths of the three boundary geodesics of the Y -piece. We shall first show that the third length ℓ_0 is uniquely determined by our triple

$$(\ell_{\mathcal{T}}(\alpha), \ell_{\mathcal{T}}(\beta), \Theta_{\mathcal{T}}(\alpha, \beta)). \quad (3.1)$$

We begin by understanding what β looks like in $\mathcal{Y}(\ell_{\mathcal{T}}(\alpha), \ell_{\mathcal{T}}(\alpha), \ell_0)$. In Figure 2 we consider our situation. The geodesic marked η has length ℓ_0 . The three black arcs joining pairs of boundary geodesics of $\mathcal{Y}(\ell_{\mathcal{T}}(\alpha), \ell_{\mathcal{T}}(\alpha), \ell_0)$ are the mutual perpendiculars. The red arc that joins the two copies of α represents β .

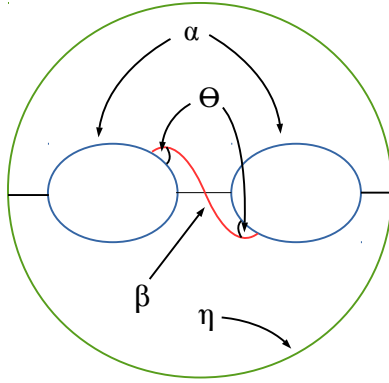


FIGURE 2. One holed torus

By a symmetry argument we obtain that the point of intersection of β and the mutual perpendicular between the two copies of α is the mid-point of both of these geodesic arcs. Now consider one of the triangles formed by two of these arcs and one of the subarcs on (one of the copies of) α . By the angle ratio formula from basic hyperbolic trigonometry:

$$\frac{\sinh \ell_p}{\sin \theta} = \frac{\sinh \frac{\ell_{\mathcal{T}}(\beta)}{2}}{\sin \frac{\pi}{2}} \Rightarrow \sinh \ell_p = \sin \theta \cdot \sinh \frac{\ell_{\mathcal{T}}(\beta)}{2} \quad (3.2)$$

where $2\ell_p$ is the length of the mutual perpendicular between the two copies of α and $(\theta) = \Theta_{\mathcal{T}}(\alpha, \beta)$. Hence ℓ_p is determined by the data (3.1).

Now recall that any Y -piece is determined (up to isometry) by the lengths of its three boundary geodesics. In particular $2\ell_p$ is a function of $\ell_{\mathcal{T}}(\alpha)$ and ℓ_0 . Fixing the value of $\ell_{\mathcal{T}}(\alpha)$ consider the map $\ell_0 \rightarrow 2\ell_p$. A simple trigonometric computation implies that this map is injective. Hence by above ℓ_0 is determined by (3.1).

Finally we get that $\mathcal{T} \setminus \alpha$ is isometric to $\mathcal{T}' \setminus \alpha'$. To conclude the theorem it suffices to observe that there is a unique way of gluing the two copies of α (or α') in the boundary of $\mathcal{Y}(\ell_{\mathcal{T}}(\alpha), \ell_{\mathcal{T}}(\alpha), \ell_0)$ (or $\mathcal{Y}(\ell_{\mathcal{T}'}(\alpha'), \ell_{\mathcal{T}'}(\alpha'), \ell_0)$) such that the red arc becomes β (or β') after the gluing. This follows from Lemma 3.2.1. \square

Now we present a proof of Theorem 1.4.1. Let us begin by recalling it.

Theorem 1.4.1 *Let S' be a closed hyperbolic surface of genus g . Let γ'_0 be a simple closed geodesic on S' and $\mathcal{P}'_0 = \{\alpha'_i : i = 1, \dots, 3g - 3\}$ be a pants decomposition of S' such that for each $i = 1, 2, \dots, 3g - 3$:*

- (1) $\ell_S(\alpha_i) = \ell_{S'}(\alpha'_i)$,
- (2) $\Theta_S(\gamma_0, \alpha_i) = \Theta_{S'}(\gamma'_0, \alpha'_i)$ and
- (3) $\Theta_S(\gamma_0, \cup_{i=1}^{3g-3} \alpha_i) = \Theta_{S'}(\gamma'_0, \cup_{i=1}^{3g-3} \alpha'_i)$

where γ_0 and $\mathcal{P}_0 = \{\alpha_i : i = 1, 2, \dots, 3g-3\}$ are as in Theorem 1.3.1. Then S' is isometric to S .

Proof. Let $\iota(\gamma_0, \cup_{i=1}^{3g-3} \alpha_i) = m = \iota(\gamma'_0, \cup_{i=1}^{3g-3} \alpha'_i)$. As one moves along γ_0 the points of intersection between γ_0 and α_i s appear one after another. Recall that the ordering of these intersections is well defined only up to the action of the cyclic permutation $(1, 2, \dots, m)$ and its inverse. For the pants decomposition obtained from Theorem 1.3.1, up to a rearrangement of the α_i s, they appear along γ_0 in the cyclic order

$$\alpha_1 \rightarrow \alpha_2 \rightarrow \dots \rightarrow \alpha_{2g-1} \rightarrow \alpha_{2g-2} \rightarrow \alpha_{2g} \rightarrow \dots \rightarrow \alpha_{3g-3} \rightarrow \alpha_2 \rightarrow \alpha_1.$$

In the first step we show that, up to the action of the cyclic permutation $(1, 2, \dots, m)$ and its inverse, the corresponding appearance of the α'_i s along γ'_0 is identical i.e.

$$\alpha'_1 \rightarrow \alpha'_2 \rightarrow \dots \rightarrow \alpha'_{2g-1} \rightarrow \alpha'_{2g-2} \rightarrow \alpha'_{2g} \rightarrow \dots \rightarrow \alpha'_{3g-3} \rightarrow \alpha'_2 \rightarrow \alpha'_1.$$

To see this let α_k appears just after α_l along γ_0 and let (ϕ_1, ϕ_2) be the ordered subset of $\Theta_S(\gamma_0, \cup_{i=1}^{3g-3} \alpha_i)$ corresponding to these two intersections. Observe that $\phi_1 \in \Theta_S(\gamma_0, \alpha_l) = \Theta_{S'}(\gamma'_0, \alpha'_l)$ and $\phi_2 \in \Theta_S(\gamma_0, \alpha_k) = \Theta_{S'}(\gamma'_0, \alpha'_k)$. Since $\Phi(\gamma'_0, \alpha'_i) \cap \Phi(\gamma'_0, \alpha'_j) = \emptyset$ for $i \neq j$ that is the only possibility as well! Hence the ordering follows.

Now let α_i, α_j and α_k bound a Y -piece in S . Then there is an ordering among α_i, α_j and α_k according to their appearance along γ_0 . In particular, up to a change of indices, we may assume that α_i appears *exactly* before α_j and α_k appear *exactly* after α_j (observe that α_i, α_j and α_k may not appear consecutively!). By the above ordering equality α'_i, α'_j and α'_k appear identically along γ'_0 . Since geodesics from a pants decomposition must intersect γ'_0 before any other geodesics do, we obtain that α'_i, α'_j and α'_k determines a Y -piece.

Recall that a hyperbolic surface can be described as a collection of marked Y -pieces and a set of relations for gluing pairs of identically marked boundary geodesics of these Y -pieces [4]. In that setting the above observation basically says that S and S' can be constructed from identical collection of Y -pieces (obtained from $S \setminus \mathcal{P}_0$ or $S' \setminus \mathcal{P}'_0$) and the gluing relations possibly differ only by twists around different geodesics (in \mathcal{P}_0 or \mathcal{P}'_0).

Next we consider a Y -piece $\mathcal{Y}(\alpha_i, \alpha_j, \alpha_k) \subset S$ with boundary geodesics α_i, α_j and α_k . Let $\mathcal{Y}'(\alpha'_i, \alpha'_j, \alpha'_k) \subset S'$ be the corresponding Y -piece with boundary geodesics α'_i, α'_j and α'_k . Since $\ell(\alpha_i) = \ell(\alpha'_i)$, $\mathcal{Y}(\alpha_i, \alpha_j, \alpha_k)$ is isometric to $\mathcal{Y}'(\alpha'_i, \alpha'_j, \alpha'_k)$ via an isometry that sends $\alpha_i \rightarrow \alpha'_i$.

Lemma 3.2.1. *Let Y be a pair of pants. Consider two boundary geodesics α_1, α_2 of Y . Let A be the collection of simple geodesic arcs in Y that joins α_1 and α_2 . Then every geodesic arc β in A is determined by the two angles β make with α_1 and α_2 , $(\Theta_Y(\beta, \alpha_1), \Theta_Y(\beta, \alpha_2))$.*

Proof. We argue by contradiction and assume that there are two such arcs β_1, β_2 such that for each $i = 1, 2$

$$\Theta_Y(\beta_1, \alpha_i) = \Theta_Y(\beta_2, \alpha_i). \quad (3.3)$$

There are two cases that need separate consideration: (i) $\beta_1 \cap \beta_2 = \emptyset$ and (ii) $\beta_1 \cap \beta_2 \neq \emptyset$. In the first case it is easy to observe that β_1 and β_2 along with subarcs of α_1 and α_2 forms a (contractible) geodesic rectangle. We reach the desired contradiction by calculating the area of this rectangle using (3.3) and the area formula 2.1. In

the second case we have a contractible triangle bounded by subarcs of β_1 and β_2 and a subarc of either α_1 or α_2 . The area calculation of this triangle using (3.3) again provides the desired contradiction. \square

Corollary 3.1. *Let T be a hyperbolic one-holed torus with geodesic boundary. Let α be a simple closed geodesic on T . Assume that we have a simple geodesic arc γ that joins two points on ∂T and intersects α exactly once. If the two angles in $\Theta_T(\gamma, \partial T)$ are identical then T has an isometry that interchanges the two points of intersection between ∂T and γ .*

Proof. Cut T along α to get the pair of pants Y and denote the two copies of α on ∂Y by α_1 and α_2 . Consider the two components of $\gamma \setminus \alpha$. Denote the component that joins ∂T and α_i by γ_i .

Observe that $(\Theta_T(\gamma_1, \partial T), \Theta_T(\gamma_1, \alpha_1)) = (\Theta_T(\gamma_2, \partial T), \Theta_T(\gamma_2, \alpha_2))$. Now Y has a rotational isometry around the midpoint of the mutual perpendicular between α_1 and α_2 . By the last lemma we conclude that γ_2 is the image of γ_1 under this isometry. \square

Now we are ready to finish the proof of Theorem 1.4.1. By the lemma above we observe that the isometry between $\mathcal{Y}(\alpha_i, \alpha_j, \alpha_k)$ and $\mathcal{Y}(\alpha'_i, \alpha'_j, \alpha'_k)$ actually sends

$$\gamma_0|_{\mathcal{Y}(\alpha_i, \alpha_j, \alpha_k)} \rightarrow \gamma'_0|_{\mathcal{Y}'(\alpha'_i, \alpha'_j, \alpha'_k)}. \quad (3.4)$$

Hence it suffices to prove that there is essentially a unique way of gluing the pairs of pants obtained from \mathcal{P}_0 or equivalently from \mathcal{P}'_0 . The above lemma (and the assumption that $\Phi(\gamma_0, \alpha_i)$ are pairwise disjoint) imply that there essentially is no choice along non-separating α_i s. Along a separating α_i there is a possibility of a twist that would interchange the two points of intersections between γ_0 and α_i . Since the ordering of appearance of α_i along γ_0 is fixed this can not happen at any but those α_i s that bound one holed torus. By the last corollary it is clear that even if there are such choices the resulting surfaces obtained from different choices are isometric. \square

4. DEHN TWIST: LENGTH AND ANGLES

For the definition of Dehn twist homeomorphisms we refer the reader to §1. This section is devoted to the understanding of the following two questions. Let α, β, γ be three simple closed geodesics on S .

Question 4.1. *How does the length of $\mathcal{D}_\alpha^n(\beta)$ grow with respect to the quantities $n, \iota(\alpha, \beta), \ell(\alpha)$ etc ?*

Question 4.2. *Is there any structural property inside an angle set and in particular in $\Theta_S(\gamma, \mathcal{D}_\alpha^n(\beta))$? If so how do they change with respect to n ?*

Of course we need to be more precise about the last question. We refer the reader to §4.3 for this.

4.1. Intersection. In our answer to these two questions we use known results on how the intersection number $\iota(\gamma, \mathcal{D}_\alpha^n(\beta))$ grow with respect to the numbers $n, \iota(\alpha, \beta), \iota(\alpha, \gamma)$ and $\iota(\beta, \gamma)$. The following three estimates are from [9, Proposition 3.2] and [14, Lemma 4.2]. Recall from the introduction that for a simple closed geodesic β and a tuple $(n_1, \dots, n_k) \in \mathbb{Z}^k$ we denote the closed geodesic freely homotopic to $D_{\alpha_1}^{n_1} \circ D_{\alpha_2}^{n_2} \cdots D_{\alpha_k}^{n_k}(\beta)$ by $T_{n_1, \dots, n_k}(\beta)$.

Proposition 4.1. *For any two simple closed geodesics β, γ :*

$$(1) \ \iota(T_\beta^n(\gamma), \gamma) = |n| \iota(\gamma, \beta)^2.$$

Let $\alpha_1, \dots, \alpha_k$ be k mutually disjoint simple closed geodesics on S . Then for any two simple closed geodesics β, γ

$$(2) \iota(T_{n_1, \dots, n_k}(\beta), \gamma) \geq \sum_{i=1}^k (|n_i| - 2) \iota(\alpha_i, \beta) \iota(\alpha_i, \gamma) - \iota(\gamma, \beta),$$

$$(3) \iota(T_{n_1, \dots, n_k}(\beta), \gamma) \leq \sum_{i=1}^k |n_i| \iota(\alpha_i, \beta) \iota(\alpha_i, \gamma) + \iota(\gamma, \beta).$$

One more topological result on the intersection number would be important in our later discussions. Let \mathcal{A} be either \mathcal{C}_α or one of the two components of $\overline{\mathcal{C}_\alpha} \setminus \alpha$. Recall that in §2 we defined end-to-end geodesic arcs in \mathcal{C}_α . These are basically simple geodesic arcs in \mathcal{C}_α that join two points one in each component of $\partial\mathcal{C}_\alpha$. We define end-to-end geodesic arcs in \mathcal{A} , when \mathcal{A} is not \mathcal{C}_α , similarly.

Lemma 4.1.1. *Let β_1, β_2 be two non-intersecting end-to-end geodesic arcs in \mathcal{A} . Then for any end-to-end geodesic arc η in \mathcal{A} one has:*

$$|\iota(\eta, \beta_1) - \iota(\eta, \beta_2)| \leq 1.$$

Proof. Without loss of generality we may assume that

$$\iota(\eta, \beta_2) \geq \iota(\eta, \beta_1) \text{ and } \iota(\eta, \beta_2) > 1.$$

Fix two points of intersection x, y between η and β_2 that occur consecutively along η and let η' be the subarc of η lying between x and y . It suffices to prove that η' intersects β_1 at least once. We argue by contradiction and assume that η' and β_1 are disjoint. Cutting \mathcal{C}_α along β_1 we obtain a rectangle $\mathcal{R}_\alpha(\beta_1)$. Since β_2 and η' do not intersect β_1 both of them are contained in $\mathcal{R}_\alpha(\beta_1)$. In particular we have a loop formed by η' and the subarc of β_2 between x and y which is contained in $\mathcal{R}_\alpha(\beta_1)$, a topological disc. This is an impossibility. \square

4.2. Lengths. Now we consider the length of $\mathcal{D}_\alpha^n(\beta)$. The next estimate is probably well-known to experts but the author was unable to locate a reference.

Proposition 4.2. *Let $\alpha_1, \alpha_2, \dots, \alpha_k$ be k mutually disjoint simple closed geodesics on S . Then for any $\beta \in \mathcal{G}(S)$ there exist non-negative integers $k_i = k_i(\alpha_i, \beta)$ such that for any $(n_1, n_2, \dots, n_k) \in \mathbb{Z}^k$ with $|n_i|$ sufficiently large one has:*

$$(1) \ell(T_{n_1, \dots, n_k}(\beta)) \leq \sum_{i=1}^k \iota(\alpha_i, \beta) |n_i| \ell(\alpha_i) + \ell(\beta)$$

$$(2) \sum_{i=1}^k \iota(\alpha_i, \beta) (|n_i| - k_i) \ell(\alpha_i) \leq \ell(T_{n_1, \dots, n_k}(\beta)).$$

Proof. For the upper bound observe that $D_{\alpha_1}^{n_1} \circ D_{\alpha_2}^{n_2} \cdots D_{\alpha_k}^{n_k}(\beta)$ is freely homotopic to the union of β and $\iota(\alpha_i, \beta) |n_i|$ copies of α_i for $i = 1, \dots, k$ after removing the points intersection properly [9]. Since $T_{n_1, \dots, n_k}(\beta)$ is the geodesic in this free homotopy class, the upper bound follows.

For the lower bound we consider the collar $\mathcal{C}_{\alpha_i} \subset S$ around α_i . Since \mathcal{C}_{α_i} and \mathcal{C}_{α_j} are mutually disjoint for $i \neq j$ it suffices to consider the length of $T_{n_1, n_2, \dots, n_k}(\beta)$ restricted to each \mathcal{C}_{α_i} . By Lemma 8.0.1 in the Appendix for any simple closed geodesic δ we have:

$$\ell(\delta|_{\mathcal{C}_{\alpha_i}}) \geq (\iota(\eta_i, \delta) - 2\iota(\alpha_i, \delta)) \ell(\alpha_i),$$

where η_i is an almost radial (see §1.1) arc in \mathcal{C}_{α_i} . So it suffices to find a simple closed geodesic whose restrictions to \mathcal{C}_{α_i} has at least one almost radial arc η_i such that

$$\iota(T_{n_1, \dots, n_k}(\beta), \eta_i) \geq \iota(\beta, \alpha_i)(|n_i| - k_i), \quad (4.3)$$

for some $k_i = k(\alpha_i, \beta)$. Observe that by a similar argument as in the first paragraph (of this proof) we can easily see that for any such geodesic arc η_i we have the upper bound

$$\iota(T_{n_1, \dots, n_k}(\beta), \eta_i) \leq \iota(\beta, \alpha_i)|n_i| + \iota(\beta, \eta_i). \quad (4.4)$$

Let γ be a geodesic on S that intersects all the α_i s for $i = 1, \dots, k$. Replacing γ by certain combination of Dehn twists of γ along α_i s, if necessary, we can assume that each sub-arc of γ in each \mathcal{C}_{α_i} is an almost radial arc. Applying Proposition 4.1(2) to γ along with Lemma 8.1.1 from the Appendix we have $n = n(\gamma, \alpha_1, \dots, \alpha_k)$ such that

$$\sum_{i=1}^k \iota(T_{n_1, \dots, n_k}(\beta)|_{\mathcal{C}_{\alpha_i}}, \gamma|_{\mathcal{C}_{\alpha_i}}) \geq \sum_{i=1}^k (|n_i| - 2)\iota(\alpha_i, \beta)\iota(\alpha_i, \gamma) - \iota(\gamma, \beta) - n. \quad (4.5)$$

To complete the proof we argue by contradiction and assume that there is a sequence $(n_{1,j}, \dots, n_{k,j})_j$ such that for any geodesic arc η appearing as subarcs of $\gamma|_{\mathcal{C}_{\alpha_1}}$ we have

$$\iota(T_{n_{1,j}, \dots, n_{k,j}}(\beta), \eta) < (|n_{1,j}| - k(n_{1,j}))\iota(\beta, \alpha_i) \quad (4.6)$$

where $k(n_{1,j}) \rightarrow \infty$ as $|n_{1,j}| \rightarrow \infty$. By (4.5) we have

$$\sum_{i=1}^k \iota(T_{n_{1,j}, \dots, n_{k,j}}(\beta)|_{\mathcal{C}_{\alpha_i}}, \gamma|_{\mathcal{C}_{\alpha_i}}) \geq \sum_{i=1}^k (|n_i| - 2)\iota(\alpha_i, \beta)\iota(\alpha_i, \gamma) - \iota(\gamma, \beta) - n.$$

Using (4.6) we get

$$\begin{aligned} & \sum_{i=2}^k \iota(T_{n_{1,j}, \dots, n_{k,j}}(\beta)|_{\mathcal{C}_{\alpha_i}}, \gamma|_{\mathcal{C}_{\alpha_i}}) - \sum_{i=2}^k (|n_i| - 2)\iota(\alpha_i, \beta)\iota(\alpha_i, \gamma) \\ & \geq (k(n_{1,i}) - 2)\iota(\alpha_i, \beta)\iota(\alpha_i, \gamma) - \iota(\gamma, \beta) - n \rightarrow \infty \end{aligned}$$

as $n_{1,i} \rightarrow \infty$. This is contradictory to by (4.4). \square

Remark 4.7. *By the last part of the proof it follows that for any γ, β and $\alpha_1, \dots, \alpha_k$ as above we have*

$$\iota(T_{n_1, \dots, n_k}(\beta)|_{\mathcal{C}_{\alpha_i}}, \gamma|_{\mathcal{C}_{\alpha_i}}) \approx |n_i|\iota(\beta, \alpha_i)\iota(\gamma, \alpha_i) \quad (4.8)$$

where the implied constant may depend on the involved geodesics.

4.3. Angles. Here we study some *structural properties* of angle sets. In the simplest case we take two simple closed geodesics α and β with $\iota(\alpha, \beta) = 1$ and consider the sequence $\beta_n = \mathcal{D}_\alpha^n(\beta)$. Our goal is to understand $\Theta_S(\beta, \beta_n)$. For our purpose it would be sufficient to understand $\Theta_S(\beta|_{\mathcal{C}_\alpha}, \beta_n|_{\mathcal{C}_\alpha})$. We begin our study by counting the number of intersections between β and β_n lying in the two components of $\mathcal{C}_\alpha \setminus \alpha$. It is reasonable to believe that these two numbers are approximately the same. Since this fact is important for us we start by giving a proof of this fact.

4.3.1. End-to-end geodesic arcs. Observe that for any simple closed geodesic β , any of its subarcs in \mathcal{C}_α is, what we called, an end-to-end geodesic arc. Recall that an end-to-end arc is a smooth simple curve on \mathcal{C}_α that are graphs of functions $[-w(\alpha), w(\alpha)] \rightarrow \mathbb{S}^1$, where we use the Fermi coordinates to identify \mathcal{C}_α with $[-w(\alpha), w(\alpha)] \times \mathbb{S}^1$. Recall further that for an end-to-end geodesic arc β in \mathcal{C}_α by $\mathcal{D}_\alpha^m(\beta)$ we denote the geodesic freely homotopic to $\mathcal{D}_\alpha^m(\beta)$ under the end point fixing homotopy.

Lemma 4.8.1. *Let ξ, η be two end-to-end geodesic arcs such that they have the same end points on $\partial\mathcal{C}_\alpha$. Then $\mathcal{D}_\alpha^m(\xi) = \eta$ for some $m \in \mathbb{Z}$.*

Proof. It is enough to show that $D_\alpha^m(\xi)$ is homotopic to η , for some $m \in \mathbb{Z}$, under the end point fixing homotopy. To show this consider any two points of intersection between ξ and η that occur consecutively along η . Consider the piecewise geodesic loop formed by the subarc of ξ and η between these two points. This loop is freely homotopic to α . Using the definition of the Dehn twists homeomorphism D_α it is not difficult to see that one of $D_\alpha(\xi)$ and $D_\alpha^{-1}(\xi)$ does not have the latter loop, up to isotopy. As a result one of them intersects η one less number of times than ξ . Repeating this procedure we get $m \in \mathbb{Z}$ such that $D_\alpha^m(\xi)$ is homotopic to η . \square

Lemma 4.8.2. *Let ξ and η be two end-to-end geodesic arcs in \mathcal{C}_α . Then the numbers of intersection between ξ and η in the two components of $\mathcal{C}_\alpha \setminus \alpha$ differ by at most two.*

Proof. We first prove the Lemma with an extra assumption that all four end points of ξ and η have the same θ coordinate equal to ψ . By Lemma 4.8.1 we know that ξ and η are Dehn twists of the ψ -radial arc η_ψ (see §1.1) of certain order i.e.

$$\mathcal{D}_\alpha^m(\eta_\psi) = \xi \text{ and } \mathcal{D}_\alpha^n(\eta_\psi) = \eta$$

for some $m, n \in \mathbb{Z}$.

Now recall the Fermi coordinates (r, θ) on \mathcal{C}_α . Using these coordinates consider the embedding of \mathcal{C}_α in \mathbb{R}^3 via the map $(r, \theta) \rightarrow (\cos \theta, \sin \theta, r)$. Recall that in the Fermi coordinates α is identified with \mathbb{S}^1 . Let $a : \mathbb{S}^1 \rightarrow \mathbb{S}^1$ be the antipodal map. Now consider the line L_ψ in \mathbb{R}^3 that intersects \mathcal{C}_α orthogonally at the points $(\cos \psi, \sin \psi, 0)$ and $(\cos a(\psi), \sin a(\psi), 0)$. The rotation of \mathbb{R}^3 by an angle of π with axis L_ψ when restricted to \mathcal{C}_α provides an isometry R_ψ of \mathcal{C}_α . It is easy to check that R_ψ interchanges the two components of $\mathcal{C}_\alpha \setminus \alpha$. Moreover, using explicit expressions one can observe that $D_\alpha^k(\eta_\psi)$ are invariant under R_ψ . By uniqueness $\mathcal{D}_\alpha^k(\eta_\psi)$ are also invariant under R_ψ for each k . It follows that the numbers of intersections between $\mathcal{D}_\alpha^m(\eta_\psi) = \xi$ and $\mathcal{D}_\alpha^n(\eta_\psi) = \eta$ in the two components of $\mathcal{C}_\alpha \setminus \alpha$ are identical.

To prove the general case we first observe that for any end-to-end geodesic arc χ whose two end points have different θ coordinates one can always find another end-to-end geodesic arc ζ disjoint from χ whose both end points have the same θ coordinates. To see this we use Lemma 4.8.1 to express χ as the Dehn twist of certain order of an almost radial arc χ_0 i.e. $\chi = \mathcal{D}_\alpha^m(\chi_0)$ for some integer m . Now by definition χ_0 does not intersect at least one radial arc say η_ϕ . Then a candidate for our ζ is $\mathcal{D}_\alpha^m(\eta_\phi)$. The general case now follows from Lemma 4.1.1. \square

4.3.2. Angle set and intersections. The fact that the two halves of $\mathcal{C}_\alpha \setminus \alpha$ contains approximately the same number of intersections between any two end-to-end geodesic arcs is not yet *visible* in their angle set. A part of our next result would make it so. Fix one set of Fermi coordinates on $\mathcal{C}_\alpha \setminus \alpha$. With respect to these coordinates consider the orientation of α from §1.3.1.

Let γ be an end-to-end geodesic arc in \mathcal{C}_α and x be the point of intersection between γ and α . Let $\pi : \mathbb{H}^2 \rightarrow S$ be a fixed universal covering such that $\pi(0) = x$ and denote the corresponding lifts of α and γ by the same alphabet. Observe that the above orientation of α corresponds to the orientation of the lift of α that decreases the height. Consider Figure 3 where the left column corresponds to situations in $\mathcal{C}_\alpha \subset S$ and the right column corresponds to one set of lifts of the involved geodesics to \mathbb{H}^2 .

Theorem 4.8.1. *Let $\Theta_S(\gamma, \alpha) = (\phi)$. Let ξ be another end-to-end geodesic arc with $\Theta_S(\gamma, \xi) = (\theta_1, \theta_2, \dots, \theta_m)$. Then:*

(1) either

$$\theta_1 > \theta_2 > \dots > \phi < \dots < \theta_{m-1} < \theta_m$$

or

$$\theta_1 < \theta_2 < \dots < \phi > \dots > \theta_{m-1} > \theta_m$$

where the angles before and after ϕ correspond to the intersections between γ and ξ that occur in the two different halves of $\mathcal{C}_\alpha \setminus \alpha$,

(2) for any $\epsilon > 0$ the cardinality $\#\{j : \theta_j \in (\phi - \epsilon, \phi + \epsilon)\} \approx m$.

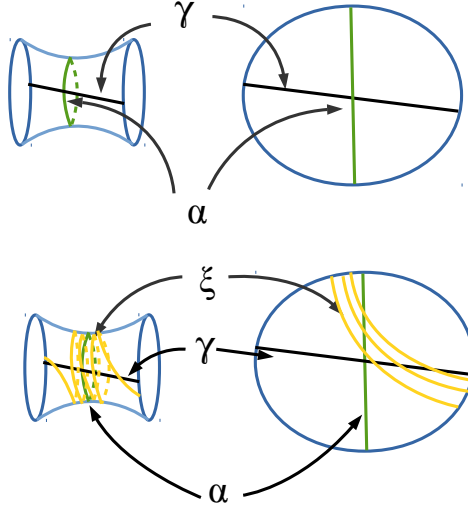


FIGURE 3. Twists and their lifts

Proof. We begin by making the observation that θ_i does not depend on the orientation of γ or ξ . Since the results we want to prove are qualitative it is enough to consider the angles corresponding to the points of intersection between γ and ξ lying in one of the halves of $\mathcal{C}_\alpha \setminus \alpha$. Without loss of generality let us consider the right half $\{(r, \theta) : 0 \leq r \leq w(\alpha)\}$ of $\mathcal{C}_\alpha \setminus \alpha$ with respect to the fixed set of Fermi coordinates on \mathcal{C}_α .

By Lemma 4.8.1 we know that γ and ξ are Dehn twists of certain order of almost radial arcs. Using Remark 2.3 we can further say that there are almost radial end-to-end geodesic arcs γ_0, ξ_0 such that for some $m_1, m_2 \in \mathbb{Z}$

$$\gamma = \mathcal{D}_\alpha^{m_1}(\gamma_0), \quad \xi = \mathcal{D}_\alpha^{m_2}(\xi_0) \quad (4.9)$$

with γ_0 and ξ_0 are either identical or disjoint. The first monotonicity appears for $m_2 - m_1 > 0$ and the second appears for $m_2 - m_1 < 0$. Observe that γ and $\mathcal{D}_\alpha^{m_1}(\xi_0)$ are either identical or disjoint. So the sign of $m_2 - m_1$, in some sense, measures

the *amount* of Dehn twists applied to ξ with respect to γ . Let us assume that $m_2 - m_1 > 0$, the other case can be dealt with similarly.

Recall that x is the point of intersection between γ and α . Let y be the point of intersection between ξ and α . Now consider the points of intersection x_1, x_2, \dots between γ and ξ arranged in the ascending order of their distances from x measured along γ (lying in the right half of $\mathcal{C}_\alpha \setminus \alpha$). For each x_i consider the subarc γ_i of γ between x_i and x and the subarc ξ_i of ξ between x_i and y . Let $\gamma_i : [0, 1] \rightarrow \mathcal{C}_\alpha$ and $\xi_i : [0, 1] \rightarrow \mathcal{C}_\alpha$ be the parametrization of γ_i and ξ_i respectively such that $\gamma_i(0) = x_i = \xi_i(0)$ and $\gamma_i(1) = x, \xi_i(1) = y$. By Lemma 8.2.1 there is a smooth homotopy $H_i : [0, 1] \times [0, 1] \rightarrow \mathcal{C}_\alpha$ between γ_i and ξ_i that has the following properties: $H_i(s, 0) = \gamma(s), H_i(s, 1) = \xi_i(s), H_i(0, t) = x_i$ and $H_i(1, \cdot)$ maps $[0, 1]$ to $\mathbb{S}^1 \simeq \alpha$. Moreover, the last map is orientation reversing with respect to the orientation of $\mathbb{S}^1 \simeq \alpha$. Lifting this homotopy to \mathbb{H}^2 we obtain lifts of ξ_i, γ and α that forms a geodesic triangle T_i . Since $H_i(1, \cdot) : [0, 1] \rightarrow \mathbb{S}^1 \simeq \alpha$ is orientation reversing, using the orientation of our fixed lift of α , we conclude that the lift of $H_i(1, \cdot)$ increases height. Making proper choices of these lifts we obtain that $T_i \subset T_{i+1}$. Hence (1) follows from the area comparison of these two triangles via (2.1).

For the second part we need to consider end-to-end geodesic arcs ξ in \mathcal{C}_α that intersects γ large number of times. Using Lemma 4.8.1 we observe that it suffices to consider end-to-end geodesic arcs of the form $(\mathcal{D}_\alpha^k(\eta))$ where η is an almost radial arc in \mathcal{C}_α . We shall first study the geodesic laminations that are obtained as limits of end-to-end geodesic arcs of this last form. Let L be one such geodesic lamination and consider the angle set $\Theta_S(\gamma, L) = (\phi_1, \phi_2, \dots)$. The structure of L is easy to describe. L consists of three leaves one of which is α . Rest of the two leaves are isolated. For each component of $\mathcal{C}_\alpha \setminus \alpha$ there is one leaf of L that spirals around α staying in this component. Since α is the only minimal component of L it follows that $\Theta_S(\gamma, L)$ has exactly one point of accumulation that corresponds to the point of intersection between γ and α i.e. ϕ . Hence we have

$$|\{\phi_i : \phi_i \notin (\phi - \epsilon, \phi + \epsilon)\}| \leq N_L(\epsilon)$$

for some integer $N_L(\epsilon)$ that a priori depends on L . To understand this dependence observe first that L is determined by two things: (i) the direction of spiraling of the two isolated leaves around α (see §2.4.2) and (ii) the two end points of L on the two components of $\partial\mathcal{C}_\alpha$. Since there are two possible directions in which the two isolated leaves of L may spiral around α it suffices to take care of these two cases separately.

Let \mathcal{L}^\pm be the collection of all those lamination (as above) whose isolated leaves respectively have \pm direction of spiraling around α . Hence any two laminations in \mathcal{L}^+ (or in \mathcal{L}^-) differ only by their end points on $\partial\mathcal{C}_\alpha$. Observe that \mathbb{S}^1 acts by isometry on \mathcal{C}_α and each component of $\mathcal{C}_\alpha \setminus \alpha$ via the θ coordinate of our Fermi coordinates. By applying appropriate amount of rotation to each leaf (to match the two end points) any lamination in \mathcal{L}^+ (or in \mathcal{L}^-) can be obtained from any other. It is not difficult to observe that the effect of these rotations on $N_L(\epsilon)$ is upper semi-continuous with respect to the angles of rotation. Hence $N_L(\epsilon)$ can be made independent of the lamination. Let us denote this uniform bound by $\mathcal{N}_\alpha(\epsilon)$. Finally for any lamination L as above and $\Theta_S(\gamma, L) = (\phi_1, \phi_2, \dots)$ we have

$$|\{\phi_i : \phi_i \notin (\phi - \epsilon, \phi + \epsilon)\}| \leq \mathcal{N}_\alpha(\epsilon). \quad (4.10)$$

Now we are ready to prove the second part of our theorem. We argue by contradiction. So we have an $\epsilon > 0$ for which we have almost radial arcs ξ_j and a sequence of integers (n_j) with $\Theta_S(\gamma, \mathcal{D}_\alpha^{n_j}(\xi_j)) = (\theta_1^j, \dots, \theta_{m_j}^j)$ such that

$$m_j - \#\{i : \theta_i^j \in (\phi - \epsilon, \phi + \epsilon)\} \rightarrow \infty. \quad (4.11)$$

Now extract a subsequence of ξ_j that converges to an end-to-end geodesic arc ξ_0 . Up to extracting subsequences, the limits of $\mathcal{D}_\alpha^{n_j}(\xi_j)$ and $\mathcal{D}_\alpha^{n_j}(\xi_0)$ are the same. Let L_0 be this limit and let $\Theta_S(\gamma, L_0) = (\phi_1^0, \phi_2^0, \dots)$. The convergence $\mathcal{D}_\alpha^{n_j}(\xi_j) \rightarrow L_0$ via (4.11) implies that

$$|\{\phi_i^0 : \phi_i^0 \notin (\phi - \epsilon, \phi + \epsilon)\}| = \infty$$

which is contradictory to (4.10) via the convergence $\mathcal{D}_\alpha^{n_j}(\xi_0) \rightarrow L_0$. \square

4.3.3. The general case. Now let γ and $\alpha_1, \dots, \alpha_k$ are as in §3.2. Let η be a simple closed geodesic which we would twist along different α_i . Assume that η intersects each α_i . Recall that $T_{n_1, \dots, n_k}(\eta)$ denotes the geodesic in the free homotopy class of $D_{\alpha_1}^{n_1} \circ \dots \circ D_{\alpha_k}^{n_k}(\eta)$. By Remark 4.7 and Lemma 4.1.1 we know that for components γ_j of $\gamma|_{\mathcal{C}_{\alpha_i}}$ and η_j of $T_{n_1, \dots, n_k}(\eta)|_{\mathcal{C}_{\alpha_i}}$ the number of intersections $\iota(\gamma_j, \eta_j) \approx |n_i|$.

Now we divide γ into different pieces $\gamma = \cup_{j=1}^l \gamma_j$ such that a γ_j is contained in either one of the \mathcal{C}_{α_i} or in the complement of all these collars. Assume that γ_j and γ_{j+1} occur consecutively along γ . So

$$\Theta_S(\gamma, T_{n_1, \dots, n_k}(\eta)) = (\Theta_S(\gamma_1, T_{n_1, \dots, n_k}(\eta)), \dots, \Theta_S(\gamma_l, T_{n_1, \dots, n_k}(\eta))).$$

Now let $\Phi(\gamma, T_{n_1, \dots, n_k}(\eta)) = \{\psi_1, \dots, \psi_p\}$ where ψ_i s are distinct. Let ϵ_0 be the minimum distance between any two distinct ψ_i, ψ_j . For each $i = 1, 2, \dots, k$ let I_i be the collection of j for which $\gamma_j \subset \mathcal{C}_{\alpha_i}$.

Theorem 4.11.1. *Assume that $n_i \rightarrow \infty$ for $i = 1, 2, \dots, k$. For $j \in I_i$ let ϕ_j be the angle of intersection between γ_j and α_i . For any $\epsilon > 0$ let $\mathcal{P}_{\phi_j}^\epsilon(\gamma, T_{n_1, \dots, n_k}(\eta))$ be the ordered subset of $\Theta_S(\gamma, T_{n_1, \dots, n_k}(\eta))$ consisting of angles in $\Theta_S(\gamma_j, T_{n_1, \dots, n_k}(\eta))$ with magnitude in $(\phi_j - \epsilon, \phi_j + \epsilon)$. Then for any $\epsilon < \epsilon_0$ one has*

$$\#\mathcal{P}_{\phi_j}^\epsilon(\gamma, T_{n_1, \dots, n_k}(\eta)) \approx \iota(\alpha_i, \eta)n_i. \quad (4.12)$$

On the other hand, for $j \notin \cup_{i=1}^k I_i$ the cardinality of $\Theta_S(\gamma_j, T_{n_1, \dots, n_k}(\eta))$ is uniformly bounded independent of (n_1, \dots, n_k) .

Proof. The first part follows from Theorem 4.8.1 and Remark 4.7. The second part follows from Lemma 8.1.1 in the Appendix. \square

Remarks 4.13. (1) *As in §1.4.2 consider a sequence $(n_{1,i}, n_{2,i}, \dots, n_{k,i})$ such that $T_{n_{1,i}, \dots, n_{k,i}}(\eta) \rightarrow L_{\text{sign}(1)\alpha_1, \dots, \text{sign}(k)\alpha_k}(\eta)$ where $\text{sign}(j)$ denotes the limiting sign of the sequence (n_i) . Observe that $\Theta_S(\gamma, \cup_{i=1}^k \alpha_i)$ is recognizable from the collection $(\Theta_S(\gamma, T_{n_{1,j}, \dots, n_{k,j}}(\eta)))_j$.*

(2) *Fix $\gamma, \alpha_1, \dots, \alpha_k$ and $\epsilon < \epsilon_0$ and consider the asymptotic in (4.12). A priori it depends on η . This dependence is uniform for $\eta \in \mathcal{N}(m_1, \dots, m_k, l)$ (see Lemma 8.1.1 in the Appendix). To see this, by Theorem 4.8.1(2), it suffices to observe that all but finitely many points of intersections between γ and $T_{n_{1,i}, \dots, n_{k,i}}(\eta)$, in a uniform way, stays inside $\cup_{i=1}^k \mathcal{C}_{\alpha_i}$. This is the statement of Lemma 8.1.1 proved in the Appendix.*

We end this section with a description of $\Theta_S(\gamma, L)$ where γ is an end-to-end arc in \mathcal{C}_α and L is a geodesic lamination in \mathcal{C}_α that has exactly two leaves one of which is α and the other, ℓ , starts at a point on $\partial\mathcal{C}_\alpha$ and spirals around α staying entirely in one of the components of $\mathcal{C}_\alpha \setminus \alpha$.

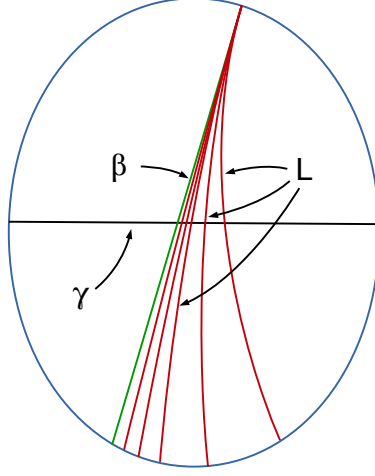


FIGURE 4. Spiral

Lemma 4.13.1. *Let $\Theta_S(\gamma, \alpha) = (\phi)$ and $\Theta_S(\gamma, L) = (\theta_1, \dots, \theta_n, \dots)$. Then the sequence (θ_i) is strictly monotone and converges to ϕ .*

Proof. Arguments here are very similar to those in the proof of Theorem 4.8.1. Recall that we have fixed one set of Fermi coordinates on \mathcal{C}_α and with respect to these coordinates there is a precise direction in which ℓ spirals around α . Assume that this direction is negative. The case of positive direction is very similar.

Taking one set of lifts of α, γ and L our current situation looks like Figure 4. To prove the monotonicity between θ_i and θ_j we compare the areas of the two triangles formed by the two lifts of L corresponding to θ_i and θ_j with the fixed lifts of α, γ . For the second part we use Theorem 2.4.1 to conclude that each point of accumulation of $\Theta_S(\gamma, L)$ corresponds to a point of intersection between γ and a minimal sub-lamination of L . Since L has exactly one minimal component, α , the only limit of (θ_i) is ϕ . \square

Remark 4.14. *Using the description of $L_{\text{sign}(1)\alpha_1, \dots, \text{sign}(k)\alpha_k}(\eta)$ from §1.4.2, the set up from Theorem 4.11.1 and the above lemma it follows that the ordered set of accumulation points of $\Theta_S(\gamma, L_{\text{sign}(1)\alpha_1, \dots, \text{sign}(k)\alpha_k}(\eta))$ is precisely $\Theta_S(\gamma, \cup_{i=1}^k \alpha_i)$.*

5. PROOF OF THEOREM 1.3.1

There are probably many different ways of finding \mathcal{P}_0 once γ_0 is chosen. We describe one such method. We shall first choose $2g - 2$ non-separating simple closed geodesics that make mutually disjoint set of angles with γ_0 and divides S into X -pieces (four holed spheres with geodesic boundary). Let us start by choosing one non-separating simple closed geodesic α_1 that intersects γ_0 exactly once. Up to applying a diffeomorphism, we may assume that γ_0 and α_1 are as in Figure 5.

Now consider another simple closed geodesic β_2 as the green curve in Figure 5. It also intersects γ_0 exactly once and does not intersect α_1 . If $\Phi(\gamma_0, \alpha_1) \neq \Phi(\gamma_0, \beta_2)$ then we choose $\alpha_2 = \beta_2$. If $\Phi(\gamma_0, \alpha_1) = \Phi(\gamma_0, \beta_2)$ then we modify β_2 as follows. Consider a simple closed geodesic η_2 , as the purple curve in Figure 5, that does not intersect α_1 and γ_0 but intersects β_2 exactly twice. Observe that $\mathcal{D}_{\eta_2}(\beta_2)$ intersects γ_0 exactly once. Moreover we have the following monotonicity.

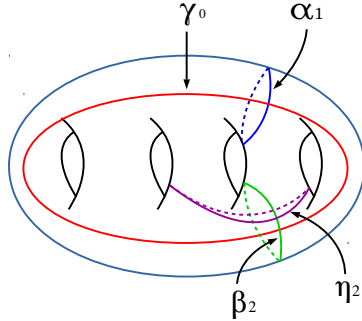


FIGURE 5. Construction of \mathcal{P}_0

Claim 5.0.1. *Let $\Theta(\gamma_0, \beta_2) = (\phi_1)$ and $\Theta(\gamma_0, \mathcal{D}_{\eta_2}(\beta_2)) = (\psi_1)$. Then $\phi_1 > \psi_1$.*

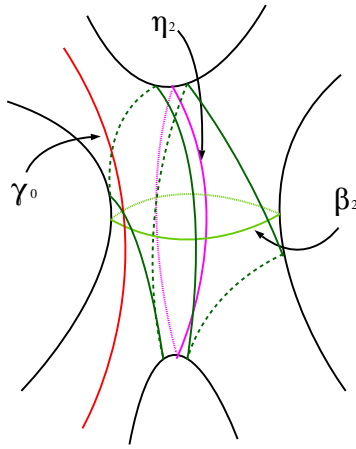


FIGURE 6. For separable geodesics

Proof. As in the proof of Theorem 4.8.1 we would lift γ_0 , β_2 and $\mathcal{D}_{\eta_2}(\beta_2)$ to \mathbb{H}^2 and compare the angles there. For that we consider the point of intersection y_0 between γ_0 and β_2 . In Figure 6 the light green curve represents β_2 , the magenta curve represents η_2 , the deep blue curve represents $\mathcal{D}_{\eta_2}(\beta_2)$ and the red arc represents an arc of γ_0 corresponding to the angle ϕ_1 . Now fix one set of lifts of γ_0 and β_2 to \mathbb{H}^2 that intersect each other at a fixed point y .

Fix one set of Fermi coordinates on \mathcal{C}_{η_2} and orient η_2 according to the orientation explained in §1.3.1 via these set of coordinates. Observe that η_2 and β_2 intersect at two points and these two points divide β_2 into two geodesic arcs one of which contains y_0 . Denote this last arc by a_{β_2} and without loss of generality assume that

this latter arc's restriction to \mathcal{C}_{η_2} is contained in the left half of $\mathcal{C}_{\eta_2} \setminus \eta_2$. Observe that η_2 and $\mathcal{D}_{\eta_2}(\beta_2)$ also intersect at two points and these two points divide $\mathcal{D}_{\eta_2}(\beta_2)$ into two geodesic arcs one of which intersects a_{β_2} . Denote this arc by $a_{\mathcal{D}_{\eta_2}(\beta_2)}$. By Lemma 8.3.1 we have parametrization of $a_{\beta_2} : [0, 1] \rightarrow S$ and $a_{\mathcal{D}_{\eta_2}(\beta_2)} : [0, 1] \rightarrow S$ of these two arcs and a smooth homotopy $H : [0, 1] \times [0, 1] \rightarrow S$ that has the following properties: $H(s, 0) = a_{\beta_2}(s)$, $H(s, 1) = a_{\mathcal{D}_{\eta_2}(\beta_2)}(s)$ and $H(0, t), H(1, t)$ maps $[0, 1]$ to η_2 . Moreover these last two maps are orientation preserving with respect to the orientation of η_2 .

To lift this homotopy to \mathbb{H}^2 we consider two lifts of η_2 as the two purple curves in Figure 7. Observe that the above orientation of η_2 provides orientations of these two geodesics. This induced orientation increases height of the left lift and decreases height for the right lift. Thus lifting H to \mathbb{H}^2 we obtain Figure 7. Now it is evident that there is a point of intersection x between the lifts \tilde{a}_{β_2} of a_{β_2} and $\tilde{a}_{\mathcal{D}_{\eta_2}(\beta_2)}$ of $a_{\mathcal{D}_{\eta_2}(\beta_2)}$.

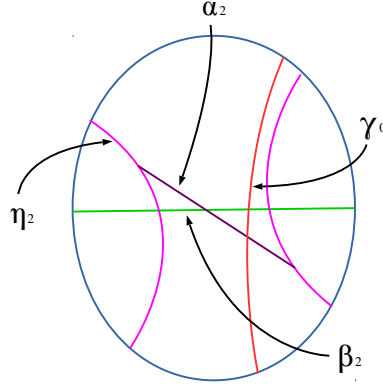


FIGURE 7. Lifted on \mathbb{H}^2

We have two cases. First, x and y are identical. In this case our claim follows from the property that H moves the two end points of \tilde{a}_{β_2} along the two lifts of η_2 along the orientation η_2 (on S). In the second case x and y are distinct points. From Figure 7 we can assume that the homotopy H between \tilde{a}_{β_2} and $\tilde{a}_{\mathcal{D}_{\eta_2}(\beta_2)}$ is a rotation around x sending \tilde{a}_{β_2} to $\tilde{a}_{\mathcal{D}_{\eta_2}(\beta_2)}$.

Observe that x divides \tilde{a}_{β_2} into two connected components. Let \tilde{b}_{β_2} be the closure of the component that contains y . So only the right lift of η_2 intersects \tilde{b}_{β_2} , say at z , and H homotopes \tilde{b}_{β_2} to a subarc $\tilde{b}_{\mathcal{D}_{\eta_2}(\beta_2)}$ of $\tilde{a}_{\mathcal{D}_{\eta_2}(\beta_2)}$. The monotonicity now follows from the positivity of the area of the triangle formed by $\tilde{b}_{\mathcal{D}_{\eta_2}(\beta_2)}$, \tilde{b}_{β_2} and the image of y under H that is a subarc of the fixed lift of γ_0 . \square

So we take $\alpha_2 = \mathcal{D}_{\eta_2}(\beta_2)$. We can repeat this procedure until we get a collection of non-separating simple closed geodesics $\alpha_1, \dots, \alpha_{2g-2}$ that divide S into a collection of X -pieces. Figure 8 explains this situation. In each of these X pieces we have the situation as in Figure 9 where the red arcs are the subarcs of γ_0 .

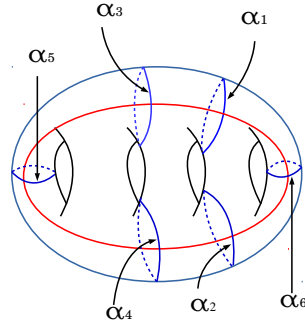


FIGURE 8. All non-separating geodesics

Here we consider two simple closed geodesics β and η as in Figure 9 where β is separating and intersects γ_0 twice and η is non-separating and do not intersect γ_0 .

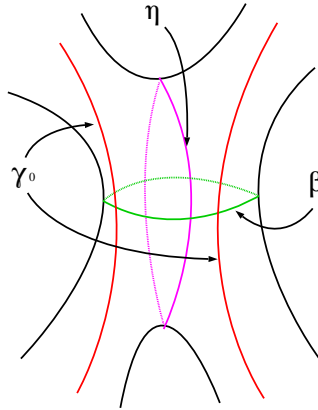


FIGURE 9. Separating geodesics

Observe that this situation is very similar to the situation for non-separating geodesics above. The only difference in this situation is that now we have two subarcs of γ_0 instead of one. Arguments in the proof of Claim 5.0.1 work for each of these two arcs. Hence sufficient number of Dehn twist along η would make sure that $\Phi(\gamma_0, \mathcal{D}_\eta^n(\beta))$ is disjoint from any finite collection of angles.

6. PROOF OF THE THEOREM 1

In this section we use the asymptotic growth of lengths and asymptotic structure of angle sets from §3 along with the theory of geodesic laminations to prove Theorem 1. So we consider two closed hyperbolic surfaces S, S' with identical length-angle spectrum. Let the genus of S and S' be g and g' respectively with $g \geq g'$.

We begin by considering a simple closed non-separating geodesic γ_0 on S and a pants decomposition $\mathcal{P}_0 = \{\alpha_i : i = 1, \dots, \alpha_{3g-3}\}$ of S provided by Theorem 1.3.1.

Now we fix a sequence $\bar{v}_n = (v_{1,n}, \dots, v_{3g-3,n}) \in \mathbb{Z}_{>0}^n$ such that $\lim_{n \rightarrow \infty} v_{i,n} = \infty$ for each i and

$$\lim_{n \rightarrow \infty} \frac{v_{i+1,n}}{v_{i,n}} = 0 \text{ for each } i = 1, \dots, 3g-2. \quad (6.1)$$

Then we consider the sequence of simple closed geodesics $T_{\bar{v}_n}(\gamma_0)$, in the free homotopy class of

$$D_{\alpha_1}^{v_{1,n}} \circ D_{\alpha_2}^{v_{2,n}} \circ \dots \circ D_{\alpha_{3g-3}}^{v_{3g-3,n}}(\gamma_0).$$

As n tends to infinity the sequence of closed geodesics $T_{\bar{v}_n}(\gamma_0)$ converge to the geodesic lamination $L_{\alpha_1, \dots, \alpha_{3g-3}}(\gamma_0)$. Since $\mathcal{L}\Theta(S) = \mathcal{L}\Theta(S')$ we have simple closed geodesics γ'_n, δ'_n on S' such that

$$(\ell_S(\gamma_0), \ell_S(T_{\bar{v}_n}(\gamma_0)), \Theta_S(\gamma_0, T_{\bar{v}_n}(\gamma_0))) = (\ell_{S'}(\gamma'_n), \ell_{S'}(\delta'_n), \Theta_{S'}(\gamma'_n, \delta'_n)).$$

A priori γ'_n depends on n . Using the standard fact that the number of simple closed geodesics on any closed hyperbolic surface with length equal to (or bounded from above by) a fixed number is finite we can assume, up to extracting a subsequence, that we have a fixed simple closed geodesic γ'_0 such that

$$(\ell_S(\gamma_0), \ell_S(T_{\bar{v}_n}(\gamma_0)), \Theta_S(\gamma_0, T_{\bar{v}_n}(\gamma_0))) = (\ell_{S'}(\gamma'_0), \ell_{S'}(\delta'_n), \Theta_{S'}(\gamma'_0, \delta'_n))$$

for some simple closed geodesics δ'_n on S' . Our goal now is to understand these simple closed geodesics δ'_n . By Remark 2.4, up to extracting a further subsequence, δ'_n converges to a geodesic lamination. We denote this geodesic lamination by L . Let $L_{\gamma'_0}$ be the smallest sub-lamination of L that contains all those leaves of L that intersect γ'_0 . Hence $\Theta_{S'}(\gamma'_0, L) = \Theta_{S'}(\gamma'_0, L_{\gamma'_0})$. By Theorem 2.4.1 we have a finite collection of minimal sub-laminations K_1, \dots, K_m whose complement in $L_{\gamma'_0}$ is a finite union of isolated leaves and each of these isolated leaves, along each of its ends, spirals around one of the K_i . Let K_1, \dots, K_l are those minimal sub-laminations of $L_{\gamma'_0}$ that intersect γ'_0 .

Lemma 6.1.1. *Each angle of intersection between γ'_0 and K_i is a point of accumulation of $\Theta_{S'}(\gamma'_0, L_{\gamma'_0})$ and every point of accumulation of $\Theta_{S'}(\gamma'_0, L_{\gamma'_0})$ is an angle of intersection between γ'_0 and one of the K_i s.*

Proof. Observe first that each K_i is also a minimal component of L . Since L is the limit of a sequence of simple closed geodesics it follows that if K_i is a simple closed geodesic then there is a leaf of L that spirals around K_i . In particular this spiraling leaf must be in $L_{\gamma'_0}$. Now we have two possible type of minimal components: not isolated and isolated. If K_i is not isolated then the lemma follows from the definition (of not isolated). If K_i is isolated then it follows from the above observation that there is at least one isolated leaf of $L_{\gamma'_0}$ that spirals around K_i . For the reverse direction observe that a point of accumulation of $\Theta_{S'}(\gamma'_0, L_{\gamma'_0})$ can not correspond to an intersection between γ'_0 and an isolated leaf. Hence we have the lemma by Theorem 2.4.1. \square

Now we use our angle sets explicitly to have deeper understanding of these K_i s. It is probably believable that if one K_i is not a simple closed geodesic then the angle sets $\Theta_S(\gamma'_0, \delta'_n)$ should look significantly different from $\Theta_S(\gamma'_0, T_{\bar{v}_n}(\gamma_0))$. For our purpose the next result would suffice.

Theorem 6.1.1. *Each of K_1, \dots, K_l is a simple closed geodesic.*

Proof. We begin by considering the angles in

$$\Phi(\gamma_0, \cup_{i=1}^{3g-3} \alpha_i) = \{\Phi_1, \dots, \Phi_M\}$$

where Φ_i s are distinct. Let ϵ_0 be the minimum of the distances between distinct Φ s. For any $\epsilon < \epsilon_0/4$ the neighborhoods $I_i^\epsilon = (\Phi_i - \epsilon, \Phi_i + \epsilon)$ are at least ϵ distance

apart. This makes sure that whenever we have an ordered subset $(\theta_1, \dots, \theta_k)$ of some $\Theta(\gamma_0, \delta_{v_n})$ that has the property that

$$|\theta_i - \theta_j| < \epsilon \text{ for all } i, j \text{ and one of the } \theta_j \in I_i^\epsilon$$

then $\theta_i \in I_i^\epsilon$ for all $i = 1, \dots, k$. Now let ϕ be an angle of intersection between γ'_0 and one of the K_i s. We shall first show that $\phi = \Phi_k$ for some k .

By the last lemma ϕ is a point of accumulation of $\Theta_{S'}(\gamma'_0, L_{\gamma'_0})$ and so we have an ordered sequence (ϕ_1, ϕ_2, \dots) of $\Theta_{S'}(\gamma'_0, L_{\gamma'_0})$ that converges to ϕ . We may choose this sequence in such a way that $\phi_j \in (\phi - \frac{\epsilon}{4}, \phi + \frac{\epsilon}{4})$ for all j . Using the convergence $\delta'_n \rightarrow L$ we have ordered subset $(\psi_1^n, \dots, \psi_{k_n}^n)$ of $\Theta(\gamma'_0, \delta'_n)$, for n sufficiently large, such that $\psi_i^n \rightarrow \phi_i$ as $n \rightarrow \infty$ (in particular the size of these ordered sets $k_n \rightarrow \infty$). We may further assume by making n larger if necessary that each $\psi_i^n \in (\phi_i - \frac{\epsilon}{2}, \phi_i + \frac{\epsilon}{2})$. This implies that

$$|\psi_i^n - \psi_j^n| < \epsilon \text{ for each } i, j = 1, \dots, k_n.$$

Using $\Theta_{S'}(\gamma'_0, \delta'_n) = \Theta_S(\gamma_0, T_{\bar{v}_n}(\gamma_0))$ and the first paragraph of this proof we conclude that each $\psi_i^n \in I_k^\epsilon$ for some k independent of i . Since we have finitely many Φ_k s, up to extracting a subsequence, we may assume that each $\psi_i^n \in I_k^\epsilon$ for some fixed k independent of i and n . Now let Φ_k be an angle of intersection between γ_0 and α_l . By Theorem 4.11.1 it follows that if we choose ϵ sufficiently small then we can make sure, up to discarding a few angles if necessary, that $(\psi_1^n, \dots, \psi_{k_n}^n)$ is an ordered subset of $\Theta(\gamma_0|_{c_{\alpha_l}}, T_{\bar{v}_n}(\gamma_0)|_{c_{\alpha_l}})$. Depending on if α_l is separating or not we have two cases. If α_l is non-separating then Φ_k is the only angle of intersection between α_l and γ_0 . If α_l is separating then we have two points of intersection between α_l and γ_0 . If both the angles at these two intersections are equal to Φ_k then again we are okay. The last situation is that the two angles are different and one of them is Φ_k . Let γ_0^1 denote $\gamma_0|_{c_{\alpha_l}}$ in the first two cases and the subarc of $\gamma_0|_{c_{\alpha_l}}$ that corresponds to the angle Φ_k in the second case. Hence $(\psi_1^n, \dots, \psi_{k_n}^n)$ is an ordered subset of $\Theta(\gamma_0^1, T_{\bar{v}_n}(\gamma_0)|_{c_{\alpha_l}})$. Using the convergence $T_{\bar{v}_n}(\gamma_0) \rightarrow L_{\alpha_1, \dots, \alpha_{3g-3}}(\gamma_0)$ and our assumption $\psi_i^n \rightarrow \phi_i$ we conclude that $\phi_i \in \Theta(\gamma_0^1, L_{\alpha_1, \dots, \alpha_{3g-3}}(\gamma_0)|_{c_{\alpha_l}})$.

From Lemma 4.13.1 we have a description for $\Theta(\gamma_0^1, L_{\alpha_1, \dots, \alpha_{3g-3}}(\gamma_0)|_{c_{\alpha_l}})$. In particular, a fixed angle can appear in $\Theta(\gamma_0^1, L_{\alpha_1, \dots, \alpha_{3g-3}}(\gamma_0)|_{c_{\alpha_l}})$ at most $2\mu(\gamma_0^1, \alpha_l)$ times and Φ_k is its only accumulation point. Let us consider an angle ψ in $\Theta(\gamma_0^1, L_{\alpha_1, \dots, \alpha_{3g-3}}(\gamma_0)|_{c_{\alpha_l}})$ that is not equal to Φ_k . Choose $\epsilon > 0$ small enough such that $(\psi - \epsilon, \psi + \epsilon)$ and $I_1^\epsilon, \dots, I_k^\epsilon$ are disjoint and (4.12) is true. Then the number of angles in any $\Theta_S(\gamma_0^1, T_{\bar{v}_n}(\gamma_0))$ that lie in $(\psi - \epsilon, \psi + \epsilon)$ is uniformly bounded, independent of n . In particular, only finitely many ϕ_i can be equal to ψ . Hence $\phi = \Phi_k$.

Now we show that each K_i contains an isolated leaf. We argue by contradiction and assume that K_i does not contain any isolated leaf. Hence each point in $\gamma'_0 \cap K_i$ is a point of accumulation of $\gamma'_0 \cap K_i$. In particular $\gamma'_0 \cap K_i$ contains uncountably many points. By the first part of this proof all the angles of these intersections must come from the finite set $\{\Phi_1, \dots, \Phi_k\}$. Let ℓ be a leaf of K_i . Since K_i is minimal ℓ must intersect γ'_0 infinitely many times. Let ℓ_0 be a subarc of ℓ between two such intersections. Using minimality once again we find subarcs ℓ_i of possibly other leaves of K_i such that $\ell_i \rightarrow \ell_0$ uniformly. Now lift the whole situation on \mathbb{H}^2 . Let γ_1, γ_2 be two fixed lifts of γ'_0 such that a lift $\tilde{\ell}_0$ of ℓ_0 joins γ_1 and γ_2 . Using the fact that $\ell_i \rightarrow \ell$ uniformly we can find lifts $\tilde{\ell}_i$ of each ℓ_i such that $\tilde{\ell}_i$ also joins γ_1 and γ_2 . Since ℓ_0 and ℓ_i s are parts of a geodesic lamination their lifts $\tilde{\ell}_0$ and $\tilde{\ell}_i$ are mutually disjoint. Thus subarcs of $\gamma_1, \gamma_2, \tilde{\ell}_i$ and $\tilde{\ell}_j$ for each $i \neq j$ bound a geodesic rectangle, say $R(i, j)$.

Now consider the angles of intersections $\Theta(\gamma_1, \tilde{\ell}_i)$. Since they must be one of Φ_1, \dots, Φ_k we can extract a subsequence of ℓ_i s such that $\Theta(\gamma_1, \tilde{\ell}_i) = (\Phi_l)$ for some l independent of i . Extracting a further subsequence we can further ensure that $\Theta(\gamma_2, \tilde{\ell}_i) = (\Phi_k)$ for some k independent of i . Finally we reach our contradiction by computing the area of $R(i, j)$ for this extracted sequence of ℓ_i, ℓ_j s (which, by our assumption, is equal to zero!) \square

Let us denote the simple closed geodesic K_i by α'_i . So there are leaves of $L_{\gamma'_0}$ that spiral around $\alpha'_1, \dots, \alpha'_l$. Of course $L_{\gamma'_0}$ can have much complicated behavior away from γ'_0 . Now we make this observation precise.

Definition 6.1.1. *Let p be a point of intersection between γ'_0 and $L_{\gamma'_0}$. Let ℓ be the leaf of $L_{\gamma'_0}$ that intersects γ'_0 at p . We say that p corresponds to a spiraling if one of the half-leaves of ℓ , determined by p , spiral around one of $\alpha'_1, \dots, \alpha'_l$ along one of its (two) ends.*

Lemma 6.1.2. *All but finitely many points of intersections between γ'_0 and $L_{\gamma'_0}$ correspond to a spiraling.*

Proof. We argue by contradiction and assume that there are infinitely many points x_1, \dots, x_n, \dots of intersections between γ'_0 and $L_{\gamma'_0}$ that does not correspond to a spiraling. Since γ'_0 is a closed geodesic the sequence of points x_1, \dots, x_n, \dots have a point of accumulation on γ'_0 . By Lemma 6.1.1 this point of accumulation must be a point of intersection between γ'_0 and a minimal component of $L_{\gamma'_0}$. By the last theorem it must be one of the α'_i s. Since α'_i is a closed geodesic all but finitely many of x_1, \dots, x_n, \dots corresponds to spiraling around α'_i . This is a contradiction. \square

Let $N_{\gamma'_0}$ be the number of points of intersection between γ'_0 and $L_{\gamma'_0}$ that does not corresponds to a spiraling. By the last lemma and Lemma 4.13.1 the (ordered) set of accumulation points of $\Theta_{S'}(\gamma'_0, L_{\gamma'_0})$ is exactly $\Theta_{S'}(\gamma'_0, \cup_{i=1}^k \alpha'_i)$. We now compare this with $\Theta_S(\gamma_0, \mathcal{P}_0)$

Lemma 6.1.3. *As ordered sets $\Theta_S(\gamma_0, \mathcal{P}_0) = \Theta_{S'}(\gamma'_0, \cup_{i=1}^k \alpha'_i)$.*

Proof. Let $\Theta_{S'}(\gamma'_0, \cup_{i=1}^k \alpha'_i) = (\theta_1, \dots, \theta_p)$. By the last lemma all but $N_{\gamma'_0}$ angles in $\Theta(\gamma'_0, L_{\gamma'_0})$ corresponds to spiraling around one of the $\alpha'_1, \dots, \alpha'_l$. Since $\Theta(\gamma'_0, L) = \Theta(\gamma'_0, L_{\gamma'_0})$ using the convergence $\delta'_n \rightarrow L$ and Lemma 4.13.1 we conclude that $\Theta_{S'}(\gamma'_0, \delta'_n)$ consists of an ordered set $(I(\theta_1), \dots, I(\theta_p))$ where each entry in $I(\theta_i)$ lie in $(\theta_i - \epsilon, \theta_i + \epsilon)$ and rest of the angles in $\Theta_{S'}(\gamma'_0, \delta'_n)$ has cardinality bounded independent of n .

Since $\Theta_{S'}(\gamma'_0, \delta'_n) = \Theta_S(\gamma_0, T_{\bar{v}_n})$ we actually know that the latter consists of an ordered set $(I(\phi_1), \dots, I(\phi_l))$ where $I(\phi_i)$ looks like Theorem 4.11.1 and the rest of the angles in $\Theta_S(\gamma_0, \delta_{v_n})$ has uniformly bounded cardinality independent of n . Comparing the two descriptions of the same set $\Theta_{S'}(\gamma'_0, \delta'_n) = \Theta_S(\gamma_0, T_{\bar{v}_n})$ we conclude the lemma. \square

Recall that L is the limit of a sequence of simple closed geodesics and there are only finitely many isolated leaves (in any geodesic lamination; Theorem 2.4.1) in L that spiral around α'_i . Hence for each i there are a finite and equal number of leaves spiraling around α'_i from both sides in the same direction. Let ξ_i be the number of leaves that spiral around α'_i from one side. Since L is the limit of (δ'_n) it follows that $\iota(\alpha'_i, \delta'_n) = \xi_i$.

Untwisting. From the above observations it is reasonable to think that, up to extracting a further subsequence, (δ'_n) are the images of a fixed simple closed geodesic $\delta'_0 \in \mathcal{G}(S')$ under various combinations of Dehn twists along α_i s. We make this precise in the next proposition.

Proposition 6.2. *There is a subsequence δ'_{m_n} of δ'_n and a simple closed geodesic δ'_0 such that*

$$\delta'_{m_n} = \prod_{i=1}^l \mathcal{D}_{\alpha'_i}^{s'_i}(\delta'_0) \quad (6.3)$$

where $\iota(\gamma', \delta'_0) \leq N_{\gamma'_0} + \sum_{i=1}^l \xi_i \cdot \iota(\gamma', \alpha'_i)$ and $N_{\gamma'_0}$ is the number from Lemma 6.1.2.

Proof. Let us start by recalling that a sub-lamination of L spirals around $\alpha'_1, \dots, \alpha'_l$. Since $\delta'_n \rightarrow L$ it follows that for n sufficiently large δ'_n has large number of twists around each α'_i . Hence applying Dehn twists to δ'_n along α'_i 's we can get simple closed geodesics that intersect γ'_0 fewer of times than δ'_n does. Following our notations from §3 for $(s_1, \dots, s_k) \in \mathbb{Z}^k$ let $T_{s_1, \dots, s_k}(\delta'_n)$ denote the simple closed geodesic freely homotopic to $D_{\alpha'_1}^{s_1} \circ \dots \circ D_{\alpha'_k}^{s_k}(\delta'_n)$. Consider a simple closed geodesic β_n such that

$$\iota(\gamma'_0, \beta_n) = \min_{(s_1, \dots, s_k) \in \mathbb{Z}^k} \iota(\gamma'_0, T_{s_1, \dots, s_k}(\delta'_n)). \quad (6.4)$$

A fairly straight forward topological argument provides that

$$\iota(\gamma'_0, \beta_n) \leq N_{\gamma'_0} + \sum_{i=1}^l \xi_i \cdot \iota(\gamma'_0, \alpha'_i). \quad (6.5)$$

Let $\delta'_n = T_{s_1^n, \dots, s_k^n}(\beta_n)$. Now estimate the number l .

Lemma 6.5.1. *The collection $\{\alpha'_i : i = 1, 2, \dots, l\}$ forms a pants decomposition \mathcal{P}' of S' . After a rearrangement of the indices*

$$\Theta(\gamma_0, \alpha_i) = \Theta(\gamma'_0, \alpha'_i).$$

Proof. The angle set $\Theta(\gamma'_0, \cup_{i=1}^l \alpha'_i)$ is the set of accumulation points of $\Theta(\gamma', L_{\gamma'})$ and by Lemma 6.1.3 we have

$$\Theta(\gamma'_0, \cup_{i=1}^l \alpha'_i) = \Theta(\gamma_0, \mathcal{P}_0).$$

Now fix one i and for α_i consider a α'_i for which $\Phi(\gamma'_0, \alpha'_i) \cap \Phi(\gamma_0, \alpha_i) \neq \emptyset$. From §4 and the last equality of angle sets we know that there are at most two choices for this.

Consider $\phi \in \Phi(\gamma'_0, \alpha'_i) \cap \Phi(\gamma_0, \alpha_i)$. On S' let ϕ be the angle of intersection between the subarc γ'_i of $\gamma'_0|_{c_{\alpha'_i}}$ and α'_i . Recall the ordered subset $\mathcal{P}_\phi^\epsilon(\gamma'_0, \delta'_n)$ of $\Theta(\gamma'_0, \delta'_n)$ that consists of angles in $\Theta(\gamma'_i, \delta'_n)$ with magnitude in $(\phi - \epsilon, \phi + \epsilon)$. By Theorem 4.11.1 and Remark 4.13 we have the asymptotic

$$\#\mathcal{P}_\phi^\epsilon(\Theta(\gamma'_0, \delta'_n)) \approx s_n^i \cdot \xi_i.$$

From our construction we know on the other hand that

$$\#\mathcal{P}_\phi^\epsilon(\gamma_0, T_{v_n}(\gamma_0)) \approx \iota(\gamma_0, \alpha_i) \cdot v_{i,n}.$$

By Lemma 6.1.2 the last two asymptotic counts are comparable i.e.

$$\iota(\gamma_0, \alpha_i) \cdot v_{i,n} \approx \xi_i \cdot s_n^i. \quad (6.6)$$

At this point we recall our choice:

$$\lim_{n \rightarrow \infty} \frac{v_{i+1,n}}{v_{i,n}} = 0 \text{ for all } i = 1, \dots, 3g-3. \quad (6.7)$$

By Lemma 6.1.3 we know that for any $\phi_1 \in \Theta_S(\gamma_0, \mathcal{P}_0)$ there is an α'_i such that $\phi_1 \in \Theta_{S'}(\gamma'_0, \alpha'_i)$. To estimate the number l we first count how many angles in $\Theta_{S'}(\gamma'_0, \cup_{i=1}^l \alpha'_i)$ can belong to the same $\Theta_{S'}(\gamma'_0, \alpha'_i)$. If $\phi_1, \phi_2 \in \Theta_{S'}(\gamma'_0, \alpha'_i)$ then by (6.6) and (6.7) it follows that there is exactly one j such that $\Theta_S(\gamma_0, \alpha_j) = (\phi_1, \phi_2)$. In particular, from the special properties of \mathcal{P}_0 from Theorem 1.3.1, it follows that at most two angles in $\Theta_{S'}(\gamma'_0, \cup_{i=1}^l \alpha'_i)$ can belong to the same $\Theta_{S'}(\gamma'_0, \alpha'_i)$ and that

happens only if they belong to one of the $\Theta_S(\gamma_0, \alpha_j)$. Hence l is at least $3g - 3$. Since $g' \leq g$ and α'_i s are mutually disjoint, we must have $g' = g$ and $l = 3g - 3$. Therefore every α_i corresponds to a unique α'_i such that $\Theta_S(\gamma_0, \alpha_i) = \Theta_{S'}(\gamma'_0, \alpha'_i)$ and we have a pants decomposition $\mathcal{P}'_0 = \{\alpha'_i : i = 1, 2, \dots, 3g - 3\}$ of S' . \square

Now we are ready to finish the proof of our proposition. It suffices to show that $\ell(\beta_n)$ is uniformly bounded. We argue by contradiction and assume that $\ell(\beta_n)$ is unbounded. In particular, there is at least one pair of pants P determined by \mathcal{P}'_0 such that the length of β_n restricted to P do not stay bounded. Now recall that for each i we have $\iota(\beta_n, \alpha'_i) = \xi_i$, a fixed finite number determined by L . Hence one of the subarcs of β_n stays entirely inside P whose length does not stay bounded. This implies that this subarc twists around one of the α'_i in ∂P a large number of times. In particular, $\iota(\gamma'_0, \beta_n)$ does not stay bounded. This is a contradiction to (6.5). \square

The only part of Theorem 1.5.1 that remains to be proven is:

Lemma 6.7.1. *After rearranging the indices according to Lemma 6.5.1 for each $i = 1, \dots, 3g - 3$ we have $\ell(\alpha'_i) = \ell(\alpha_i)$.*

Proof. Recall that our geodesic $T_{\bar{v}_n}(\gamma_0)$ is the geodesic freely homotopic to $D_{\alpha_1}^{v_1, n} \circ \dots \circ D_{\alpha_{3g-3}}^{v_{3g-3}, n}(\gamma_0)$. Thus we have the following length comparison from Theorem 4.2

$$\begin{aligned} \sum_{i=1}^{3g-3} \iota(\gamma_0, \alpha_i) \cdot (v_{i,n} - k_i) \cdot \ell(\alpha_i) &\leq \ell(T_{\bar{v}_n}(\gamma_0)) \\ &\leq \ell(\gamma_0) + \sum_{i=1}^{3g-3} \iota(\gamma_0, \alpha_i) \cdot v_{i,n} \cdot \ell(\alpha_i) \end{aligned} \quad (6.8)$$

where k_i are some fixed positive integers depending on $\alpha_1, \dots, \alpha_{3g-3}, \gamma_0$. By the last proposition we also know that δ'_{m_n} is the geodesic freely homotopic to $\mathcal{D}_{\alpha_1}^{s_1} \circ \dots \circ \mathcal{D}_{\alpha_{3g-3}}^{s_{3g-3}}(\delta'_0)$ which provides via Theorem 4.2

$$\sum_{i=1}^{3g-3} \xi_i \cdot (s_n^i - k'_i) \cdot \ell(\alpha'_i) \leq \ell(\delta'_{m_n}) \leq \ell(\delta'_0) + \sum_{i=1}^{3g-3} \xi_i \cdot s_n^i \cdot \ell(\alpha'_i) \quad (6.9)$$

where k'_i are fixed some positive integers depending on $\alpha'_1, \dots, \alpha'_{3g-3}, \delta'_0$. The rest of the arguments consist of computing some limits using: (1) the equality $\ell(\delta_{v_n}) = \ell(\delta'_n)$, (2) the inequalities (6.8) and (6.9) and (3) the asymptotic behavior (6.6). For example, to prove $\ell(\alpha'_1) = \ell(\alpha_1)$ we use (6.6) to find that

$$\lim_{n \rightarrow \infty} \frac{\ell(\delta_{v_{m_n}})}{\iota(\gamma_0, \alpha_1) \cdot v_{1, m_n}} = \lim_{n \rightarrow \infty} \frac{\ell(\delta'_{m_n})}{\xi_1 \cdot s_{m_n}^1}.$$

Using (6.7) we observe that the left limit is $\ell(\alpha_1)$ by (6.8) and the right limit is $\ell(\alpha'_1)$ by (6.9). Now we use induction and assume $\ell(\alpha_i) = \ell(\alpha'_i)$ for $i \leq k-1$. Using (6.6) once again we obtain the equality

$$\lim_{n \rightarrow \infty} \frac{\ell(\delta_{v_{m_n}}) - \sum_{i=1}^{k-1} \iota(\gamma_0, \alpha_i) \cdot v_{i, m_n} \ell(\alpha_i)}{\iota(\gamma_0, \alpha_k) \cdot v_{k, m_n}} = \lim_{n \rightarrow \infty} \frac{\ell(\delta'_{m_n}) - \sum_{i=1}^{k-1} \xi_i \cdot s_{m_n}^i \ell(\alpha'_i)}{\xi_k \cdot s_{m_n}^k}.$$

As above, using (6.7) we can observe that by (6.8) the left limit is $\ell(\alpha_k)$ and by (6.9) the right limit is $\ell(\alpha'_k)$. \square

7. FURTHER REMARKS AND QUESTIONS

From the methods and already known results the following questions arise naturally.

7.1. Enlarged length-angle spectrum: Since the marked length spectrum determines the metric on S_g among all negatively curved metrics ([18], [8]) it is tempting to conjecture a similar generalization for Theorem 1. We would like to point out that such a result is false. To see this we first recall that for any fixed closed hyperbolic surface S there is an open set $U_S \subset S$ such that no simple closed geodesic on S enter U_S [2], [5]. Perturbing the metric inside U_S it is possible to construct a negatively curved surface S' that is not isometric to S but has length-angle spectrum identical to S . It is, however, non-trivial to answer the following questions.

One may consider all closed geodesics on S and the corresponding length-angle data:

$$\mathcal{L}'\Theta(S) = \{(\ell_\gamma, \ell_\delta, \Theta(\gamma), \Theta(\delta), \Theta(\gamma, \delta)) : \gamma, \delta \in \mathcal{C}(S)\} \quad (7.1)$$

where $\Theta(\gamma) = \Theta(\gamma, \gamma)$. Then a natural question to ask is if $\mathcal{L}'\Theta(S)$ determines the metric among all metrics on a fixed closed surface? A more ambitious question would be to ask if $\mathcal{L}'\Theta(S)$ determines the metric among all metrics on all closed orientable surfaces?

7.2. Simple length spectrum rigidity: It is mentioned in the introduction that Wolpert's method shows that there are only finitely many closed hyperbolic surface of fixed genus that has the same simple length spectrum. An interesting question to ask: Does the simple length spectrum determine the surface?

Due to the works [21], [20] and many other authors it is known that the length spectrum is not rigid for higher genus hyperbolic surfaces. The constructions in these papers involve the whole length spectrum and so it seems difficult to find a counterexample to our question following these methods.

8. APPENDIX

In this small section we explain some basic results used in the paper that are probably known to experts.

8.1. Lengths of end-to-end arcs. The first result is about the length of an end-to-end arc inside the collar \mathcal{C}_α around α . Fix a set of Fermi coordinates (r, θ) on \mathcal{C}_α .

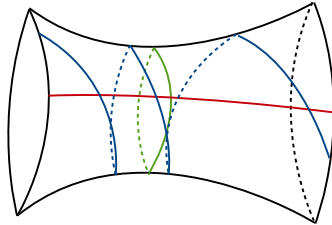


FIGURE 10. Length in a collar

Lemma 8.0.1. *Let γ be an end-to-end geodesic arc in \mathcal{C}_α around α . Let η be an almost radial arc in \mathcal{C}_α . Then $\ell(\gamma) \geq (\iota(\gamma, \eta) - 2)\ell(\alpha)$.*

Proof. Let η_ϕ be a radial arc that does not intersect η . By Lemma 4.1.1 we know that

$$\iota(\gamma, \eta_\phi) \geq \iota(\gamma, \eta) - 1. \quad (8.1)$$

Consider a subarc γ' of γ between two consecutive intersections with η_ϕ . The projection from this subarc to α via the θ coordinate (of the Fermi coordinates) is surjective. Since this projection is length decreasing we have $\ell(\gamma') \geq \ell(\alpha)$. We obtain the lemma by adding up all the pieces of γ between its consecutive intersections with η_ϕ via (8.1). \square

8.2. Uniform bound on the number of intersections. Our next result is used in §3 where we study how the length and angle sets evolve under various Dehn twists. Let $\alpha_1, \dots, \alpha_k$ be k mutually disjoint simple closed geodesics. Let γ be a simple closed geodesic that intersects each α_i . We consider the sets

$$\begin{aligned} \mathcal{N}(m_1, \dots, m_k, l) &= \{\eta \in \mathcal{GL}(S) : \iota(\eta, \alpha_i) = m_i \text{ for } i = 1, 2, \dots, k \\ &\text{and } \iota(\eta, \gamma) \leq l\}. \end{aligned}$$

Our purpose here is to understand how the intersections between γ and various Dehn twists of η along $\alpha_1, \dots, \alpha_k$ are located on S .

Lemma 8.1.1. *There is $N = N(m_1, m_2, \dots, m_k, l) > 0$ such that for any $\eta \in \mathcal{N}(m_1, \dots, m_k, l)$ and any tuple $(n_1, n_2, \dots, n_k) \in \mathbb{Z}^k$ one has*

$$\iota(\gamma|_{S \setminus \cup_{i=1}^k \mathcal{C}_{\alpha_i}}, T_{n_1, n_2, \dots, n_k}(\eta)) \leq N.$$

Proof. Let $\eta \in \mathcal{N}(m_1, \dots, m_k, l)$. Let $(n_{1,j}, n_{2,j}, \dots, n_{k,j}) \in \mathbb{Z}^k$ be a sequence such that $T_{n_{1,j}, n_{2,j}, \dots, n_{k,j}}(\eta) \rightarrow L$ a geodesic lamination. The structure of L is easy to describe. The minimal sub-laminations of L are those of η and $\alpha_1, \dots, \alpha_k$. Since $\iota(\eta, \gamma) \leq l$ no non-isolated minimal component of η intersect γ . In particular, the only non-isolated minimal components of L that intersect γ are $\alpha_1, \dots, \alpha_k$. Hence we can find a $N = N(L)$ such that $\iota(\gamma|_{S \setminus \cup_{i=1}^k \mathcal{C}_{\alpha_i}}, L) \leq N$.

To prove the lemma we argue by contradiction and assume that we have $\eta_j \in \mathcal{N}(m_1, \dots, m_k, l)$ and a sequence $(p_{1,j}, p_{2,j}, \dots, p_{k,j}) \in \mathbb{Z}^k$ be such that for all j

$$\iota(\gamma|_{S \setminus \cup_{i=1}^k \mathcal{C}_{\alpha_i}}, T_{p_{1,j}, p_{2,j}, \dots, p_{k,j}}(\eta_j)) \geq j. \quad (8.2)$$

Up to extracting a subsequence both (η_j) and $(T_{p_{1,j}, p_{2,j}, \dots, p_{k,j}}(\eta_j))$ converge to some geodesic laminations. It is not that difficult to see that if $\eta_j \rightarrow \eta$ then the limit of $T_{p_{1,j}, p_{2,j}, \dots, p_{k,j}}(\eta_j)$ up to extracting correct subsequences is the same as the limit of $T_{p_{1,j}, p_{2,j}, \dots, p_{k,j}}(\eta)$ which we denote by L_0 . Now $\eta \in \mathcal{N}(m_1, \dots, m_k, l)$ so by the first paragraph of this proof we have a $N_0 = N_0(L_0)$ such that

$$\iota(\gamma|_{S \setminus \cup_{i=1}^k \mathcal{C}_{\alpha_i}}, L_0) \leq N_0.$$

On the other hand, from (8.2) and the convergence $T_{p_{1,j}, p_{2,j}, \dots, p_{k,j}}(\eta_j) \rightarrow L$, we have

$$\iota(\gamma|_{S \setminus \cup_{i=1}^k \mathcal{C}_{\alpha_i}}, L_0) = \infty.$$

Hence we have a contradiction. \square

8.3. Dehn twist and homotopy. In the proofs of Theorem 4.8.1 and Theorem 1.3.1 we have used certain qualitative facts about Dehn twists. To recall the scenario let α be a simple closed geodesic on S and let \mathcal{C}_α be the collar neighborhood around α . Fix a set of Fermi coordinates on \mathcal{C}_α and orient α according to the orientation explained in §1.3.1.

Let η, ξ be two end-to-end geodesic arcs. By Theorem 4.8.1 there is another end-to-end geodesic arc ξ_0 and $m \in \mathbb{Z}$ such that $\xi = \mathcal{D}_\alpha^m(\xi_0)$ with η and ξ_0 are either identical or disjoint. Clearly $\iota(\eta, \xi) = m$. Let x be the point of intersection between α and η and y be the point of intersection between α and ξ . Consider the right half R of $\mathcal{C}_\alpha \setminus \alpha$ with respect to the starting Fermi coordinates. Let x_1, \dots, x_k be the points of intersection between η and ξ that lies in R arranged in the ascending order of their distances from x along η . Let η_i be the subarc of η between x and x_i and ξ_i be the subarc of ξ between y and x_i . Let $\eta_i : [0, 1] \rightarrow R$ and $\xi_i : [0, 1] \rightarrow R$ be their parametrization such that $\eta_i(0) = x_i = \xi_i(0)$ and $\eta_i(1) = x, \xi_i(1) = y$.

Lemma 8.2.1. *There is a smooth homotopy $H : [0, 1] \times [0, 1] \rightarrow R$ between η_i and ξ_i such that: $H(s, 0) = \eta_i(s), H(s, 1) = \xi_i(s)$ and $H(1, t) : [0, 1] \rightarrow \alpha$ is orientation reversing.*

Proof. In a sense Figure 11 is our complete proof. The semi-annular regions are fundamental domains for \mathcal{C}_α . In these fundamental domains we can explicitly draw lifts of any end-to-end geodesic arc. Namely, for $\mathcal{D}_\alpha^m(\xi_0)$ we would consider the two end points of ξ_0 . Then we would use explicit expression for $\mathcal{D}_\alpha^m(\xi_0)$ to draw one of its explicit lifts in $|m|$ consecutive fundamental domains of \mathcal{C}_α .

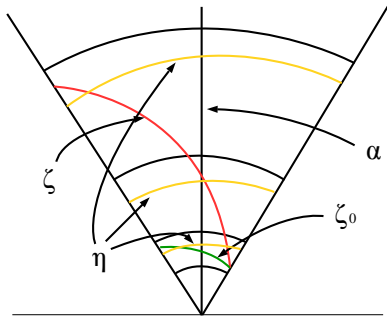


FIGURE 11. Homotopy I

Finally to draw a lift of $\mathcal{D}_\alpha^m(\xi_0)$ explicitly we would recall that the latter is the geodesic (there is exactly one such) that joins the end points of the last lift of $\mathcal{D}_\alpha^m(\xi_0)$. \square

Remark 8.3. *If we consider the left half of $\mathcal{C}_\alpha \setminus \alpha$ instead of the right half, with respect to the starting Fermi coordinates, then the last lemma for the corresponding subarcs needs the modification that the map $H(1, t) : [0, 1] \rightarrow \alpha$ is orientation preserving, instead of orientation reversing.*

Now we consider another Dehn twist considered in the proof of Theorem 1.3.1. To explain our situation let us consider an X -piece. Let β, η be the arcs as in picture. Consider the collar \mathcal{C}_η around η and fix a set of Fermi coordinates in it. Consider the orientation of η determined by these coordinates as in §1.3.1. Observe that β and $\mathcal{D}_\eta(\beta)$ are divided into two geodesic arcs by η . We shall consider the left half of these two arcs (determined by the Fermi coordinates). Let us denote these two arcs by a_β and $a_{\mathcal{D}_\eta(\beta)}$.

Lemma 8.3.1. *There are parametrization $a_\beta : [0, 1] \rightarrow X, a_{\mathcal{D}_\eta(\beta)} : [0, 1] \rightarrow X$ and a smooth homotopy $H : [0, 1] \times [0, 1] \rightarrow X$ such that $H(s, 0) = a_\beta(s), H(s, 1) = a_{\mathcal{D}_\eta(\beta)}(s)$ and $H(0, t), H(1, t)$ are smooth maps from $[0, 1] \rightarrow \eta$. Moreover the last two maps are orientation preserving.*

Proof. Consider the hyperbolic funnel $\mathbb{T}_\eta = \mathbb{H}^2 / \langle \gamma_\eta \rangle$ where γ_η is a generator of $\pi_1(\mathcal{C}_\eta) \subset \pi_1(X)$. In particular, we can lift β and $\mathcal{D}_\eta(\beta)$ on \mathbb{T}_η . Observe that the Fermi coordinates on \mathcal{C}_η provides a coordinate system on \mathbb{T}_η . With respect to these extended coordinates we consider cylindrical neighborhoods of η that are bounded by curves that look like $\{r = c\}$, called equidistant curves. These neighborhoods are very similar to collar neighborhoods. In particular, we can replace \mathcal{C}_α by one of these neighborhoods in the last lemma to get a similar result.

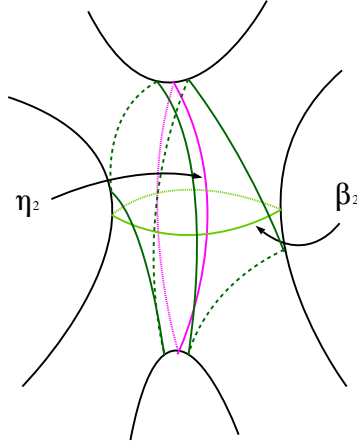


FIGURE 12. Homotopy II

Let \mathcal{C} be the smallest such cylindrical neighborhood that contains the four intersections between the lifts of β and $\mathcal{D}_\eta(\beta)$ closest from η . Since Dehn twist is defined up to homotopy, it follows that each subarc of $\mathcal{D}_\eta(\beta)$ in \mathcal{C} is the Dehn twist of a subarc of β under the end point fixing homotopy. The existence of our homotopy now follows from a modified version of the last lemma. \square

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