THÈSE

En vue de l'obtention du

DOCTORAT DE L'UNIVERSITÉ DE TOULOUSE

Délivré par
Université Toulouse III Paul Sabatier (UT3 Paul Sabatier)

Discipline ou spécialité
Mathématiques Fondamentales

Présentée et soutenue par
Sugata MONDAL

le 29 octobre 2013

Titre

Small eigenvalues of hyperbolic surfaces

École doctorale

ED MITT : Mathématiques Fondamentales

Unité de recherche

Institut de Mathématiques de Toulouse - UMR 5219

Directeurs de thèse

Jean-Pierre OTAL — Directeur de recherche, Institut de Mathématiques de Toulouse

Rapporteurs

Peter BUSER — Professeur honoraire, École Polytechnique Fédérale de Lausanne

Gilles COURTOIS — Directeur de recherche, Université Pierre et Marie Curie

Autres membres du jury

Dominique BAKRY — Professeur, Institut de Mathématiques de Toulouse
Greg MCSHANE — Professeur, Université Joseph Fourier
Xavier BUFF — Professeur, Institut de Mathématiques de Toulouse

Introduction

A hyperbolic surface is a two dimensional complete Riemannian manifold of constant sectional curvature -1. Such a surface is isometric to a quotient $\mathbb{H}/\Gamma$, where $\mathbb{H}$ is the Poincaré upper half-plane and Γ is a discrete and torsion free subgroup of $\mathrm{PSL}(2, \mathbb{R})$. Each hyperbolic surface carries a second order *elliptic* differential operator Δ, called the Laplace operator or the Laplacian, that acts on the space of functions of the surface. In this thesis we consider hyperbolic surfaces with finite area. When S is closed, the spectrum of Δ consists of a discrete set:

$$0 = \lambda_0(S) < \lambda_1(S) \leq ... \leq \lambda_i(S) \leq ...$$

where $\lambda_i(S)$ denotes the i-th eigenvalue of S and a number in the sequence is repeated according to its multiplicity as eigenvalue. In this case the set of eigenvalues does not have a finite point of accumulation and so $\lambda_n(S) \to \infty$ as $n \to \infty$. The number $\frac{1}{4}$ has a special significance in the *spectral theory* of the Laplacian of hyperbolic surfaces. Eigenvalues of Δ below $\frac{1}{4}$ are called *small eigenvalues*. Existence of hyperbolic surfaces with small eigenvalues was originally shown by B. Randol [R1] using the famous trace formula of A. Selberg. P. Buser uses a different approach and shows that for any $g \geq 2$:

Theorem 0.0.1. ***(Buser [Bu, Chapter 8])***
Given $\delta > 0$ there exists a closed hyperbolic surface S of genus g such that $\lambda_{2g-2}(S) < \delta$.

He also showed that for a fixed genus, the number of small eigenvalues has a topological upper bound. More precisely, he proved the following theorem:

Theorem 0.0.2. ***(Buser [Bu, Chapter 8])***
For any closed hyperbolic surface S of genus g: $\lambda_{4g-2}(S) > \frac{1}{4}$.

It was however not known until recently whether this upper bound is sharp. It was conjectured that $2g - 2$ is an upper bound. This is proved for genus two surfaces by P. Smutz [Sch] and for arbitrary genus by Otal-Rosas [O-R]. The result of Otal-Rosas is actually stated for any hyperbolic surfaces with finite area.

Theorem 0.0.3. ***(Otal-Rosas)***
Any finite area hyperbolic surface of type (g, n) has at most $2g - 2 + n$ small eigenvalues.

Motivated by this result, in this thesis, we try to understand the relation between small eigenvalues and topology of a hyperbolic surface. More precisely, our aim is to obtain bounds on the number of small eigenvalues in terms of the topology of the surface.

In Chapter I we review some known properties of eigenvalues and eigenfunctions of the Laplace operator on hyperbolic surfaces. We discuss two eigenvalue problems: the *closed eigenvalue problem* and the *Dirichlet eigenvalue problem*. We review two variational characterizations of these eigenvalues: *Rayleigh's theorem* and the *min-max principle*. Then we recall two geometric inequalities, the *Faber-Krahn inequality* and the *Cheeger's inequality*, that will be used in Chapter 2. In the second part we review some results on *nodal sets* and *nodal domains* of eigenfunctions of the Laplacian. In particular, we recall a theorem of S. Y. Cheng [Ch] that describes the nodal set around a point where the gradient of the eigenfunction vanishes. Then we review a lemma of J. P. Otal [O] that provides topological description of nodal sets and nodal domains of small eigenfunctions. After that we recall two *local* bounds for eigenfunctions of the Laplacian that will be used in Chapter 3. We end the chapter by giving Fourier expansions for eigenfunctions on two hyperbolic surfaces: cylinders and cusps.

Chapter II begins with the construction of P. Buser that proves Theorem 0.0.1. Then we focus on the Otal-Rosas result Theorem 0.0.3. This result provides a global lower bound on the $2g-2$-th eigenvalue for closed hyperbolic surfaces of genus g. First section of this chapter is devoted to closely follow the proof of Theorem 0.0.3 and provide the following geometric lower bound:

Theorem 2.1.4 *Let S be a closed hyperbolic surface of genus g and let $s(S)$ denote the systole of S. Then $\lambda_{2g-2}(S) > \frac{1}{4} + \epsilon_0(S)$ where $\epsilon_0(S)$ can be taken any quantity smaller than*

$$min\{\frac{1}{4(g-1)}, \frac{1}{4}((\frac{\cosh\rho_0}{\sinh\rho_0})^2 - 1)\}$$

where $2s(S)\sinh\rho_0 = |S|$.

In the next section we consider the i-th eigenvalue, λ_i, as a function of the metric i.e. λ_i is viewed as a function on the moduli space $\mathcal{M}_g$. We present a result concerning limiting behavior of λ_i as metrics converges to $\partial\mathcal{M}_g$. In the last section, using similar ideas as in the proof of Theorem 2.1.4, we prove a result that provides a lower-bound for certain *anti symmetric* cuspidal eigenvalue of certain noncompact hyperbolic surfaces of finite area.

In Chapter III we consider a sequence of hyperbolic surfaces (S_m) that converges to $S_\infty \in \overline{\mathcal{M}_{g,n}}$. When $n \neq 0$, the spectrum of the Laplacian on these surfaces consist of two parts: the *continuous spectrum* and the *discrete spectrum*. The discrete spectrum is again composed of two parts: the *residual spectrum* and *cuspidal spectrum*. The cuspidal spectrum is arranged in the ascending order and the i-th cuspidal eigenvalue of S is denoted by $\lambda_i^c(S)$. If S does not have i many cuspidal eigenvalues then we put $\lambda_i^c(S) = \infty$. We consider a sequence (λ_m, ϕ_m) of eigenpairs of S_m such that $\lambda_m \to \lambda_\infty < \infty$. Behavior of the sequence of eigenfunctions (ϕ_m) is then studied in the special case when, for each m, (λ_m, ϕ_m) is a small cuspidal eigenpair. We recall four results: [C-C], [He], [Ji] and [Wo] that concern this type of limiting. Motivated by these results we prove the following theorem:

Theorem 3.2.1 *Let $S_m \to S_\infty$ in $\overline{\mathcal{M}_{g,n}}$. Let (λ_m, ϕ_m) be a normalized (L^2-norm of ϕ_m is 1) small cuspidal eigenpair of S_m. Assume that λ_m converges to λ_∞. Then one of the following holds:*
(1) *There exist strictly positive constants ϵ, δ such that* $\limsup \|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$. *Then, up to extracting a subsequence, (ϕ_m) converges to a λ_∞-eigenfunction ϕ_∞ of S_∞.*
(2) *For each $\epsilon > 0$ the sequence $(\|\phi_m\|_{S_m^{[\epsilon,\infty)}}) \to 0$. Then, up to extracting a subsequence, (ϕ_m) converges to the zero function on S_∞. Moreover, there exist constants $K_m \to \infty$ such that, up to extracting a subsequence, $(K_m\phi_m)$ converges to a linear combination of Eisenstein series and (possibly) a cuspidal λ_∞-eigenfunction of S_∞.*
The later possibility arises only when: $S_\infty \in \partial\mathcal{M}_{g,n}$ and $\lambda_\infty = \frac{1}{4}$.

We apply this result to give a new and elementary proof of a result of D. Hejhal.

In Chapter IV we study small cuspidal eigenfunctions of finite area hyperbolic surfaces. In particular, we try to understand the following conjecture of Jean-Pierre Otal and Eulalio Rosas on the maximum possible number of such eigenvalues [O-R].

Conjecture *For any $S \in \mathcal{M}_{g,n}$: $\lambda_{2g-3}^c(S) > \frac{1}{4}$.*

We then use Theorem 3.2.1 from Chapter III and a topological property of nodal sets of small eigenfunctions to provide the following:

Theorem 4.1.1 $\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$ *is an unbounded, open subset of* $\mathcal{M}_{g,n}$.

Here

$$\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1) = \{S \in \mathcal{M}_{g,n} : \lambda^c_{2g-1}(S) > \frac{1}{4}\}.$$

In Chapter V, with the help of continuity properties of λ_i and a construction of P. Buser, we recall a proof of the fact that each λ_i is bounded. Then we focus on λ_1 and ask the question if the maximum of λ_1 over $\mathcal{M}_g$ is more than $\frac{1}{4}$ or not. Results due to Burger-Buser-Dodziuk [BBD] and Brooks-Makover [B-M] say that there exist surfaces with λ_1 arbitrary close to $\frac{1}{4}$. However, the surfaces constructed in these results are not of the same genus. Using topological arguments, as in Chapter 2, we prove that the answer is yes in the case of genus two i.e. there are surfaces in $\mathcal{M}_2$ such that $\lambda_1 > \frac{1}{4}$. Moreover, we prove that the subset of $\mathcal{M}_2$ containing surfaces with $\lambda_1 > \frac{1}{4}$ disconnects $\mathcal{M}_2$.

Contents

Chapter 1

Eigenvalues and eigenfunctions of the Laplacian of hyperbolic surfaces

This chapter is divided into three sections. In the first section we begin from the definition of the Laplace operator (or the Laplacian) on hyperbolic surfaces. In the second section we consider two eigenvalue problems related to the Laplacian and recall two variational characterizations of these eigenvalues: *Rayleigh's theorem* and the *min-max principle.* Then we recall two geometric inequalities that provide lower bounds for these eigenvalues. In the third section we consider eigenfunctions of the Laplacian. We recall the description, due to S. Y. Cheng [Ch], of *nodal sets* of eigenfunctions of the Laplacian on surfaces. Next we recall a topological property of nodal sets and nodal domains of *small eigenfunctions* of finite area hyperbolic surfaces due to J. P. Otal [O]. Then we recall two results that provide local bounds for eigenfunctions of the Laplacian. In the end we recall Fourier expansions of eigenfunctions on two hyperbolic surfaces: cusps and cylinders.

1.1 The Laplacian on hyperbolic surfaces

The Poincaré upper half-plane $\mathbb{H}$ is $\{x + iy \in \mathbb{C} : y > 0\}$ with the Riemannian metric:

$$ds^2 = \frac{dx^2 + dy^2}{y^2}.$$

This metric is called the hyperbolic metric. The Laplacian of this metric is given by the formula

$$\Delta = -y^2\{\frac{\partial^2}{\partial^2 x} + \frac{\partial^2}{\partial^2 y}\}.$$

Each element of the group $\mathrm{PSL}(2, \mathbb{R})(= \mathrm{SL}(2, \mathbb{R})/(\pm I))$ acts on $\mathbb{H}$ by *Möbius transformation*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} z \to \frac{az + b}{cz + d}. \tag{1.1}$$

This action is by orientation preserving isometries of $\mathbb{H}$. A *hyperbolic surface* is a two dimensional Riemannian manifold which is isometric to a quotient $\mathbb{H}/\Gamma$, where Γ is a *Fuchsian group*, i.e. a discrete torsion-free subgroup of PSL(2,$\mathbb{R}$). The action of PSL(2,$\mathbb{R}$) leaves the Laplacian Δ invariant in the sense that for each $\gamma \in$ PSL(2, $\mathbb{R}$): $\Delta(f \cdot \gamma) = (\Delta f) \cdot \gamma$. Thus Δ induces an operator on the quotient $S = \mathbb{H}/\Gamma$. This operator is called the Laplacian on S. We shall use the symbol Δ for this Laplacian also. Now we consider two specific types of hyperbolic surfaces: cylinders and cusps.

1.1.1 Cylinders

A *hyperbolic cylinder* $\mathcal{C}$ *with core geodesic* γ is the quotient $\mathbb{H}/ < \tau >$ where τ is a *hyperbolic isometry* of $\mathbb{H}$ and γ is the image of the *axis of* τ under the quotient map. Let $l_\gamma = 2\pi l$ denote the length of γ i.e. the hyperbolic distance $d_{\mathbb{H}}(x, \tau x)$ for some x on the axis of τ. We will use the following *Fermi coordinates* on $\mathcal{C}$. First choose an orientation of γ. Then parametrize γ with constant speed equal to l and identify γ with $\mathbb{R}/2\pi\mathbb{Z}$. Now choose an orientation of the normal bundle of γ in $\mathcal{C}$. The Fermi coordinates, with all these orientations understood, assign to each point $p \in \mathcal{C}$ the pair $(r, \theta) \in \mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$ where r is the signed distance of p from γ and θ is the projection of p on γ [Bu, p. 4]. These coordinates give a diffeomorphism of this hyperbolic cylinder to $\mathbb{R} \times \mathbb{R}/2\pi\mathbb{Z}$. In terms of these coordinates the hyperbolic metric is given by:

$$ds^2 = dr^2 + l^2 \cosh^2 r d\theta^2.$$

The Laplacian is given by the formula:

$$-\left(\frac{d^2}{dr^2} + \tanh r \frac{d}{dr} + \frac{1}{l^2 \cosh^2 r}\frac{d^2}{d\theta^2}\right).$$

For $w \geq l$ we define the *collar* $\mathcal{C}^w$ around γ by

$$\mathcal{C}^w = \{(r, \theta) \in \mathcal{C} : l_\gamma \cosh r < w,\ 0 \leq \theta \leq 2\pi\}.$$

Then $\mathcal{C}^w$ is diffeomorphic to an annulus whose each boundary component has length w. Now let S be a hyperbolic surface and let γ be a simple closed closed geodesic on S. The *Collar Theorem* of Linda Keen [K] says that $\mathcal{C}^1$ around γ embeds in S (more precisely, $\mathcal{C}^{w(l_\gamma)}$ embeds in S where $w(l_\gamma) = l_\gamma \cosh(\sinh^{-1}(\frac{1}{\sinh \frac{l_\gamma}{2}})) > 1$ and $w(l_\gamma) \approx 2$).

1.1.2 Cusps

Denote by ι the parabolic isometry $\iota : z \to z + 2\pi$. The quotient $\mathbb{H}/ < \iota >$ is called a cusp $\mathcal{P}$ at ∞. For $t > 0$ the half infinite annulus $\{x + iy : y > \frac{2\pi}{t}, 0 \leq x \leq 2\pi\}/ < \iota >$ is denoted by $\mathcal{P}^t$ and will also be called a cusp. The boundary curve $\{y = \frac{2\pi}{t}\}$ is a *horocycle* of length t that we identify with $\mathbb{R}/t\mathbb{Z}$ by arc-length parametrization. One can parametrize $\mathcal{P}^t$ using the *horocycle coordinates* [Bu, p. 4] with respect to its boundary horocycle. The horocycle coordinates assigns to a point $p \in \mathcal{P}^t$ the pair $(r, \theta) \in \mathbb{R}_{\geq 0} \times \{\mathbb{R}/t\mathbb{Z}\}$ where r is the distance of p from the horocycle $\{y = \frac{2\pi}{t}\}$ and θ the projection of p on the horocycle. In terms of these coordinates hyperbolic metric takes the form:

$$ds^2 = dr^2 + (\frac{t}{2\pi})^2 e^{-2r} d\theta^2.$$

We shall use the (x, y) coordinates on $\mathcal{P}^t$ more frequently.

A noncompact hyperbolic surface S with finite area has finitely many *punctures*. Recall that the cusp $\mathcal{P}^1$ (in fact $\mathcal{P}^2$) around each puncture of S embeds in S and that those cusps corresponding to distinct punctures have disjoint interiors (ref. [Bu, Chapter 4]). We call them *standard cusp*s. Observe that the area and boundary length of a standard cusp is equal to 1. For $t \leq 1$ denote the disjoint union $\bigcup_{c \in S} \mathcal{P}^t$ by $S_c^{(0,t)}$ where c ranges over distinct cusps in S.

1.2 Eigenvalues of the Laplacian

Let S be a hyperbolic surface and Δ be the Laplacian on S.

Definition 1.2.1. *Let $\lambda > 0$ be a real number and $f \in C^\infty(S) \cap L^2(S)$ be a nonzero function on S. The pair (λ, f) is called an eigenpair of S if $\Delta f = \lambda f$. One calls λ and f an eigenvalue and an eigenfunction respectively (sometimes a λ-eigenfunction). If $\lambda \leq \frac{1}{4}$ then λ, f and (λ, f) are respectively called small eigenvalue, small eigenfunction and small eigenpair.*

Two Eigenvalue Problems

The *closed eigenvalue problem* is posed for a closed hyperbolic surface S. The problem is to find all eigenvalues of S i.e. to find all possible $\lambda \in \mathbb{R}$ such that there exists a nonzero, smooth function f on S such that

$$\Delta f = \lambda f. \tag{1.2}$$

For non-compact hyperbolic surfaces without boundary we shall consider the same problem and require furthermore $f \in L^2(S)$.

Next, let N be a hyperbolic surface. Let $S \subset N$ be a sub-surface of N such that $\overline{S}$ is compact, ∂S is non-empty and piecewise smooth. The second type of problem is to find all possible $\lambda \in \mathbb{R}$ such that there exists a nonzero, smooth function f on S which is continuous on $\overline{S}$ such that

$$\begin{cases} \Delta f = \lambda f \text{ on } S \\ f \equiv 0 \text{ on } \partial S. \end{cases} \tag{1.3}$$

This problem is referred to as the *Dirichlet boundary value problem* for S. We shall consider the same problem for those S which have smooth boundary but $\overline{S}$ is not compact in N also. In this case we require that $f \in L^2(S)$.

We recall the following result on existence of eigenvalues. So S is either closed or a subsurface of a hyperbolic surface N that satisfies the conditions to pose the Dirichlet eigenvalue problem.

Theorem 1.2.2. *[Cha, p-8] For both the eigenvalue problems, the set of eigenvalues of S consists of a sequence*

$$0 \leq \lambda_0(S) \leq \lambda_1(S) \leq ... \leq \lambda_i(S) \leq ...$$

where $\lambda_i(S)$ denotes the i-th eigenvalue of S and each number in the sequence is repeated according to its multiplicity as eigenvalue. The sequence $(\lambda_i(S))$ does not have any finite limit point. Moreover, the eigenspaces corresponding to distinct eigenvalues are orthogonal in $L^2(S)$ and $L^2(S)$ is spanned by the direct sum of the eigenspaces.

Green's formulas

Let S be a hyperbolic surface and let $\mu_{\mathbb{H}}$ be the hyperbolic area measure of S. For vector fields X and Y on S consider the inner-product

$$< X, Y >= \int_S < X_p, Y_p >_p d\mu_{\mathbb{H}}.$$

Let $\mathcal{F}_c(S)$ denote the space of all smooth vector fields on S that has compact support. Denote by $\mathcal{L}^2(S)$ the closure of $\mathcal{F}_c(S)$ with respect to the above inner-product. Then $\mathcal{L}^2(S)$ is a Hilbert space whose elements are measurable vector fields X with

$$\|X\|^2 = \int_S |X_p|^2 d\mu_{\mathbb{H}} < \infty.$$

For a C^1-function f we de note by ∇f the gradient of f.

Green's Formula 1. Let $h \in C^1(S)$ and $f \in C^2(S)$ be such that $h(\nabla f)$ has compact support in S. If $\partial S = \emptyset$ then

$$\int_S h\Delta f d\mu_{\mathbb{H}} = \int_S < \nabla h, \nabla f > d\mu_{\mathbb{H}}.$$

Green's Formula 2. Let $h \in C^1(\overline{S})$ and $f \in C^2(\overline{S})$ be such that $h(\nabla f)$ has compact support in $\overline{S}$. Then

$$\int_S h\Delta f d\mu_{\mathbb{H}} = \int_S < \nabla h, \nabla f > d\mu_{\mathbb{H}} - \int_{\partial S} h\nu(f) d\mu_l$$

where μ_l denotes the length measure of ∂S.

Given $f \in L^2(S)$ we say that $Y \in \mathcal{L}^2(S)$ is a *derivative of f in the sense of distribution* if for any C^1 vector filed X with compact support one has:

$$< Y, X >= - < f, divX > .$$

In case such a Y exists, it is unique, and we denote it by ∇f. We shall denote by $\mathcal{H}(M)$ the space of all $f \in L^2(M)$ which has derivative in the sense of distribution. On this space we have the inner-product

$$< f, g >_1 = < f, g >_{L^2(S)} + < \nabla f, \nabla g >$$

with norm

$$\|f\|_1^2 = \|f\|_{L^2(S)}^2 + \|\nabla f\|^2.$$

It is known that for each $f \in \mathcal{H}(S)$: $\|f\|_1 < \infty$. When ∂S is sufficiently smooth $C^\infty(S)$ is dense in $\mathcal{H}(S)$.

Let ϕ be an eigenfunction for the closed eigenvalue problem on S and let $f \in \mathcal{H}(S)$. Then one has the following consequence of the **Green's formula 1**:

$$< \Delta\phi, f >=< \nabla\phi, \nabla f > .$$

Now let S be a sub-surface of hyperbolic surface N such that $\overline{S}$ is compact, ∂S is non-empty and piecewise smooth. Let $C_0^\infty(S)$ denote the space of all smooth functions f on S such that f has compact support and $f|_{\partial S}$ the restriction of f to ∂S is zero. Let $\mathcal{H}_0(S)$ be the closure of $C_0^\infty(S)$ in $\mathcal{H}(S)$ with respect to $\|.\|$-norm. Now let ϕ be an eigenfunction of the Dirichlet eigenvalue problem on S and let $f \in \mathcal{H}_0(S)$. Then one has the following consequence of **Green's formula 2**:

$$< \Delta\phi, f >=< \nabla\phi, \nabla f > .$$

1.2.3 Variational estimates

In this subsection we recall some variational characterizations of eigenvalues of the two eigenvalue problems discussed in 1.2. Let S be either a closed hyperbolic surface or a subsurface of hyperbolic surface N that satisfies the conditions to pose the Dirichlet eigenvalue problem. For $f \neq 0 \in \mathcal{H}(S)$ one considers, $\mathcal{R}(f)$, the Rayleigh quotient of f:

$$\mathcal{R}(f) = \frac{\|\nabla f\|^2}{\|f\|^2}.$$

Theorem 1.2.4. ***(Rayleigh's Theorem** [Cha, p-16])*
Closed case: *Let S be a closed hyperbolic surface. For the closed eigenvalue problem on S: $\lambda_0(S) = 0$ and*

$$\lambda_1(S) = inf_{f \in \mathcal{H}(S) \setminus \{0\}, \int_S f dV = 0} \mathcal{R}(f).$$

For $i \geq 1$ denote by H_i the subspace of $\mathcal{H}(\mathcal{S})$ spanned by the first i eigenfunctions. Let $\perp H_i$ denote the L^2-orthogonal complement of H_i in $L^2(S)$. Then for $i \geq 2$:

$$\lambda_i(S) = inf_{f \in \perp H_i} \mathcal{R}(f).$$

Dirichlet case: *Let S be a sub-surface of a hyperbolic surface N such that $\overline{S}$ is compact, ∂S is non-empty and piecewise smooth. For the Dirichlet eigenvalue problem on S:*

$$\lambda_0(S) = inf_{f \in \mathcal{H}_0(S)} \mathcal{R}(f).$$

For $i \geq 1$ denote by H_i^0 the subspace of $\mathcal{H}_0(S)$ spanned by the first i eigenfunctions. Let $\perp H_i^0$ denote the L^2-orthogonal complement of H_i^0 in $\mathcal{H}_0(S)$. Then for $i \geq 1$:

$$\lambda_i(S) = inf_{f \in \perp H_i^0} \mathcal{R}(f).$$

Another variational characterization for eigenvalues of the closed eigenvalue problem is the following:

Theorem 1.2.5. ***(Min-max Principle)***
Let S be a closed hyperbolic surface. For the closed eigenvalue problem on S one has:

$$\lambda_i(S) = \inf_{V : \dim V = i} \sup_{f \in V \setminus \{0\}} \mathcal{R}(f) \tag{1.4}$$

where V is a subspace of $L^2(S)$.

1.2.6 Two geometric inequalities

We recall two geometric inequalities, the *Faber-Krahn inequality* and the *Cheeger's inequality*, that provide lower bounds for the first eigenvalue of the Dirichlet eigenvalue problem. Let S be a sub-surface of a hyperbolic surface N such that $\overline{S}$ is compact and ∂S is piecewise smooth. Recall that $\lambda_0(S)$ denotes the first Dirichlet eigenvalue of S.

To each open set $\Omega \subset \mathbb{H}$, consisting of a finite union of disjoint *regular* domains (with piecewise smooth boundary) in S, associate the geodesic disc D in $\mathbb{H}$ with center i and satisfying:

$$\mu_{\mathbb{H}}(\Omega) = \mu_{\mathbb{H}}(D) \tag{1.5}$$

where $\mu_{\mathbb{H}}$ denotes the area measure of $\mathbb{H}$. Let μ_l denote the length measure of $\mathbb{H}$. Isoperimetric inequality for $\mathbb{H}$ says that for any Ω in $\mathbb{H}$, equality in (1.5) implies:

$$\mu_l(\partial\Omega) \geq \mu_l(\partial D) \tag{1.6}$$

with equality if and only if Ω is isometric to D. Therefore we have the following theorem due to Faber and Krahn.

Theorem 1.2.7. ***(Faber-Krahn)***[Cha, p-87] *For any Ω in $\mathbb{H}$ let D be the geodesic disc described as above. Then one has the following inequality:*

$$\lambda_0(\Omega) \geq \lambda_0(D) \tag{1.7}$$

with equality in (1.7) *if only if Ω is isometric to D.*

Now we recall the Cheeger's inequality. We consider a noncompact hyperbolic surface S possibly having nonempty boundary and possibly having compact closure.

Definition 1.2.8. *The Cheeger's constant of S, $h(S)$, is defined by*

$$h(S) = \inf_{\Omega} \frac{\mu_l(\partial\Omega)}{\mu_{\mathbb{H}}(\Omega)}$$

where Ω ranges over all open sub-surfaces of S, with compact closure and smooth boundary in S.

Theorem 1.2.9. *(**Cheeger**)[Cha, p-95]*

$$\lambda_0(S) \geq \frac{h^2(S)}{4}. \tag{1.8}$$

For the closed eigenvalue problem on a closed hyperbolic surface S, a similar inequality, also called the Chheger's inequality, exists. However, in that case, the definition of the Cheeger's constant is different from the above one [Cha].

1.3 Eigenfunctions of the Laplacian

In this section we recall some properties of eigenfunctions of the Laplacian. Let S be a closed hyperbolic surface and let $(\lambda_i(S), \phi_i)$ denote the sequence of all eigenpairs of S where $\lambda_i(S)$ denote the i-th eigenvalue of S (see Theorem 1.2.2).

Definition 1.3.1. *Let $f : S \to \mathbb{R}$ be a smooth function. The nodal set $\mathcal{Z}(f)$ of f is defined as:*

$$\mathcal{Z}(f) = \{x \in M : f(x) = 0\}.$$

Components of $S \setminus \mathcal{Z}(f)$ are called nodal domains of f.

Recall that c is called a *regular value* of f if $\nabla_x f \neq 0$ for all $x \in f^{-1}(c)$. Let f be a finite linear combinations of ϕ_i's. Therefore f is smooth (by Theorem 1.2.2) and, by Sard's theorem, almost all values of f are regular. For $x \in \mathcal{Z}(f)$ if $\nabla_x f \neq 0$ then, by implicit function theorem, $\mathcal{Z}(f)$ is an 1 dimensional submanifold in a neighborhood of x. In a neighborhood of a point $y \in \mathcal{Z}(f)$ where $\nabla_y f = 0$ the geometry of $\mathcal{Z}(f)$ is not so simple. However, when f is an eigenfunction of the Laplacian, the following theorem of S.Y. Cheng [Ch] describes $\mathcal{Z}(f)$ near such a point.

Theorem 1.3.2. *(**Cheng**) Let S be a 2-dimensional manifold with a C^∞ metric. Then, for any solution of the equation $(\Delta + h(x))f = 0$, $h \in C^\infty(S)$, one has:*
(i) Critical points on the nodal set $\mathcal{Z}(f)$ are isolated.
(ii) Any critical point in $\mathcal{Z}(f)$ has a neighborhood N in S which is diffeomorphic to the disc $\{z \in \mathbb{C} : |z| < 1\}$ by a C^1- diffeomorphism that sends $\mathcal{Z}(f) \cap N$ to an equiangular system of rays.

Now we recall the famous theorem of Courant, known as the **Courant's nodal domain theorem**, that provides a global bound on the number of nodal domains. For two dimensional manifolds this theorem says the following:

Theorem 1.3.3. *(**Courant's nodal domain theorem**)*
For each of the eigenvalue problems the number of nodal domains of an eigenfunction corresponding to the i-th eigenvalue is at most $i + 1$.

Corollary
(1) *For the closed eigenvalue problem an eigenfunction corresponding to the first nonzero eigenvalue has exactly two nodal domains.*
(2) *For the Dirichlet eigenvalue problem each eigenfunction corresponding to the zeroth eigenvalue has constant sign on S and so the multiplicity of λ_1 is exactly one. Moreover an eigenfunction corresponding to the first eigenvalue has exactly two nodal domains.*

Local description of the nodal set of a finite linear combination of eigenfunctions does not follow from Cheng's theorem. In Chapter 2 we will have to consider such a sum in the case when the metric on S is analytic. We recall the following theorem which would be helpful in this situation.

Theorem 1.3.4. ***(analyticity)***
Let S be a surface with an analytic metric. Then eigenfunctions of S are real analytic functions in the interior of S.

Therefore if f is a finite linear combination of eigenfunctions then f is analytic and its nodal set is the zero set of a real analytic function. Then Proposition 5 in [O-R] says that $\mathcal{Z}(f)$ is the union of a locally finite graph without free vertex and possibly a set of isolated points.

Topological properties of small eigenfunctions

Let S be hyperbolic surface without boundary. Recall that any eigenvalue of S below $\frac{1}{4}$ is called a small eigenvalue.

Definition 1.3.5. *An open subset $\Omega \subset S$ (resp. a graph $G \subset S$) is called incompressible if the fundamental group of any connected component of Ω (resp. of G) maps injectively into $\pi_1(S)$.*

We will use the following lemma about nodal set and nodal domains of small eigenfunctions [O, Lemma 1].

Lemma 1.3.6. ***(Otal)*** *Let S be a hyperbolic surface. Let $0 < \lambda \leq \frac{1}{4}$ and $f : S \to \mathbb{R}$ be a λ-eigenfunction. Then the graph $\mathcal{Z}(f)$ is incompressible and the Euler characteristic of each nodal domain f is negative.*

In Chapter II we shall prove an extension of this lemma in the particular case when S is closed.

1.3.7 Local bounds for eigenfunctions

Here we recall two properties of eigenfunctions of the Laplacian of hyperbolic surfaces. These properties will be used in Chapter 3. We begin with the mean value property of J. D. Fay [F].

Let $D_r \subset \mathbb{H}$ be a geodesic disc around a point z_0. Let $\Delta f = s(s-1)f$ in D_r. Then f has the following mean value property [F, Corollary 1.3].

Theorem 1.3.8.

$$f(z_0) = \frac{1}{m(r,s)} \int_{D_r} f(z) d\mu_{\mathbb{H}}, \tag{1.9}$$

where $m(r,s)$ is a continuous function of r and s only which has the following asymptotic:

$$m(r,s) \sim \pi r^2 \quad as \quad r \to 0.$$

Next we recall a bound on gradients of eigenfunctions of the Laplacian depending on its L^p-norms. Let $\Omega \subset \mathbb{H}$ be a bounded open set. We consider the following norms on the space of smooth functions on $\overline{\Omega}$:

$$(i)\ \|\phi\|_{L^r(\Omega)} = \left(\int_{\Omega} |\phi|^r dx\right)^{\frac{1}{r}} \text{ for } r > 1,$$

$$(ii)\ \|\nabla\phi\|_{\Omega} = \sup_{\Omega}|\nabla\phi(x)| \tag{1.10}$$

The following theorem is a particular case of the L^p Schauder interior estimate for solutions of elliptic differential equations [BJS, p-235, Theorem 4].

Theorem 1.3.9. *Let S be a hyperbolic surface. Let (λ, u) be an eigenpair of S. Let Ω be a bounded domain in $\mathbb{H}$ and let Ω_0 be a compact subset of Ω. Then there exists a constant $C < \infty$, depending only on Ω, Ω_0 and a bound on λ, such that*

$$\|\nabla u\|_{\Omega_0} \leq C\Big(\|u\|_{L^p(\Omega)} + \|u\|_{C^0(\Omega)}\Big). \tag{1.11}$$

1.3.10 Fourier expansions of eigenfunctions on cylinders and cusps

Let $\mathcal{C}$ be the cylinder with core geodesic γ that has length $2\pi l = l_\gamma$. Denote (r, θ) the Fermi coordinates in $\mathcal{C}$ (see 1.1). Any smooth function f on $\mathcal{C}$ can be expressed as a Fourier series in the θ-coordinate:

$$f(r,\theta) = a_0(r) + \sum_{j=1}^{\infty}\Big(a_j(r)\cos j\theta + b_j(r)\sin j\theta\Big), \tag{1.12}$$

the convergence being uniform over compact sets. The functions $a_j = a_j(r)$ and $b_j = b_j(r)$ are defined on $\mathbb{R}$ and the pair (a_j, b_j) is called the *j-th Fourier coefficients of f*. When f is a λ-eigenfunction, a_j and b_j are solutions of the differential equation in the r variable:

$$\frac{d^2\phi}{dr^2} + \tanh r\frac{d\phi}{dr} + (\lambda - \frac{j^2}{l^2\cosh^2 r})\phi = 0. \tag{1.13}$$

The change of variable $u(r) = \cosh^{\frac{1}{2}}(r)\phi(r)$ transforms (3.15) into

$$\frac{d^2u}{dr^2} = \Big((\frac{1}{4} - \lambda) + \frac{1}{4\cosh^2 r} + \frac{j^2}{l^2\cosh^2 r}\Big)u. \tag{1.14}$$

Let s_j (resp. c_j) be the solution of (3.20) satisfying the conditions: $s_j(0) = 0$ and $s_j^{'}(0) = 1$ (resp. $c_j(0) = 1$ and $c_j^{'}(0) = 0$). Since (3.20) is invariant under $r \to -r$ one has: $s_j(-r) = -s_j(r)$ and $c_j(-r) = c_j(r)$ for all $j \geq 0$. Therefore there exists $t > 0$ such that $s_j > 0$ and $c_j^{'} > 0$ on $(0, t]$. We will not need the explicit form of these solutions. For our purpose a convexity property (see Chapter 3) will suffice.

Let $\mathcal{P}$ be a cusp at ∞. Let (x, y) be the coordinates on $\mathcal{P}$ induced from the upper half-plane. Any smooth function f on $\mathcal{P}$ can be expressed as a Fourier series in the x coordinate as follows:

$$f(x,y) = f_0(y) + \sum_{j=1}^{\infty}\Big(f_j(y)\sin jx + g_j(y)\cos jx\Big), \tag{1.15}$$

the convergence being uniform over compact sets. The pair (f_j, g_j) is defined on $\mathbb{R}^+$ and are called the j-th Fourier coefficient of f. When f is a λ-eigenfunction f_j and g_j are the two linearly independent solutions of the differential equation:

$$\frac{d^2\phi}{dy^2} + \left(\frac{\lambda}{y^2} - j^2\right)\phi = 0. \tag{1.16}$$

Let $\lambda = s(1-s)$ for $s = \frac{1}{2} + ir$ with $r \in \mathbb{R}$. In the case $j = 0$ the two linearly independent solutions of (1.16) are explicit:

$$\frac{1}{2}(y^s + y^{1-s}) \text{ and } \frac{1}{2s-1}(y^s + y^{2s-1}) \text{ if } s \neq \frac{1}{2}$$

$$\sqrt{y} \text{ and } \sqrt{y}\log y \text{ if } s = \frac{1}{2} \tag{1.17}$$

For $j \geq 1$ the two linearly independent solutions are:

$$\sqrt{\frac{2jy}{\pi}} K_{s-\frac{1}{2}}(jy) \text{ and } \sqrt{2jy\pi} I_{s-\frac{1}{2}}(jy) \tag{1.18}$$

where $K_\nu(y)$ and $I_\nu(y)$ are the standard Bessel functions [I, Appendix B.4]. They have the following asymptotic behavior:

$$\sqrt{\frac{2jy}{\pi}} K_{s-\frac{1}{2}}(jy) \sim e^{-jy} \text{ and } \sqrt{2jy\pi} I_{s-\frac{1}{2}}(jy) \sim e^{jy} \tag{1.19}$$

Therefore, if one assumes that f is bounded by some power of y as $y \to \infty$ then one has:

$$f(x,y) = f_0(y) + \sum_{j=1}^{\infty} f_j \sqrt{jy} K_{s-\frac{1}{2}}(jy) \cos jx + g_j \sqrt{jy} K_{s-\frac{1}{2}}(jy) \sin jx. \tag{1.20}$$

Using the complex variable z the above Fourier expansion can also be expressed in terms of the *Whittaker function*, $W_s(z)$, as [I, Proposition 1.5]:

$$f(z) = f_0(y) + \sum_{j \neq 0} f_j^{'} W_s(jz). \tag{1.21}$$

Chapter 2

Geometric lower bounds on eigenvalues of hyperbolic surfaces

We begin by recalling a construction of Peter Buser [Bu] that shows the existence of closed hyperbolic surfaces with small eigenvalues. Then we state the Otal-Rosas [O-R] result on the number of small eigenvalues of hyperbolic surfaces of finite area that confirms a conjecture of Buser. In the next section, closely following the proof of the Otal-Rosas theorem in the case of closed hyperbolic surfaces, we obtain a quantitative version of their result in Theorem 2.1.4. In the following section we consider the i-th eigenvalue λ_i as a function on $\mathcal{M}_g$ and obtain a limiting behavior for it. This behavior distinguishes between $i \leq 2g-3$ and $i \geq 2g-2$. In the last section, using arguments similar to those used in the proof of Theorem 2.1.4, we obtain a bound on the number of *anti symmetric* cuspidal eigenvalues for certain noncompact hyperbolic surfaces of finite area.

2.1 Hyperbolic surfaces with small eigenvalues

Existence of surfaces with small eigenvalues was proved originally by B. Randol [R1] using the famous trace formula of A. Selberg. We begin by recalling another constructive method, due to P. Buser [Bu], that shows that such surfaces exists.

Theorem 2.1.1. *(**Buser**) For any $\epsilon > 0$ there exist surfaces $S \in \mathcal{M}_g$ such that $\lambda_{2g-3}(S) < \epsilon$.*

Now we quickly describe the construction. Consider a genus g hyperbolic surface M admitting a pair of pants decomposition $P_1, ..., P_{2g-2}$ such that the boundary geodesics of each P_i has length $\frac{\epsilon}{6}$. Observe that surfaces with this property can be constructed explicitly using the Fenchel-Nielsen coordinates [Bu, Chapter 6]. Now, for each $1 \leq i \leq 2g-2$, consider the function f_i:

$$f_i(x) = \begin{cases} d(x, \partial P_i) \ \textbf{ if } \ d(x, \partial P_i) \leq 1 \ \textbf{ and } \ x \in P_i \\ 1 \ \textbf{else where in } \ P_i. \end{cases} \tag{2.1}$$

Extend f_i on the rest of M by zero. Then $\{f_i\}_{i=1}^{2g-2}$ is an orthogonal set of functions because their supports do not overlap. One may check that the Rayleigh quotients of f_i, $\mathcal{R}(f_i) < \epsilon$. Using the min-max principle (see Chapter 1) one concludes that M has at least $2g-2$ many eigenvalues below ϵ.

The above construction extends to hyperbolic surfaces with finite area. So there exist finite area hyperbolic surfaces of type (g, n) that has $2g-2+n$ small eigenvalues. For closed hyperbolic surfaces Buser shows the following.

Theorem 2.1.2. *(**Buser**) A closed hyperbolic surface of genus g can have at most $4g-2$ many small eigenvalues.*

In the general case i.e. for a hyperbolic surface with finite area and type (g, n) it was conjectured that $2g-2+n$ should be the maximal number. For closed hyperbolic surfaces of genus two this conjecture is confirmed by P. Schmutz [Sch]. In [O-R] Jean-Pierre Otal and Eulalio Rosas proves this conjecture in the general case.

Théorèm 0.2 ([O-R]) *For any $S \in \mathcal{M}_{g,n}$ the number of small eigenvalues of S is at most $2g-2+n$ i.e. $\lambda^c_{2g-2+n}(S) > \frac{1}{4}$.*

In view of the above theorem it is clear that, for a closed hyperbolic surface S of genus g, $\frac{1}{4}$ works as a lower bound for $\lambda_{2g-2}(S)$. We shall closely follow the proof of [O-R, Théorèm 0.2], for closed hyperbolic surfaces, to prove the following theorem which gives a geometric lower bound on λ_{2g-2}. We first give the definition of *systole* for a surface.

Definition 2.1.3. *Systole of a hyperbolic surface S is the infimum of lengths of closed geodesics on S.*

Theorem 2.1.4. *Let S be a closed hyperbolic surface of genus g and let $s(S)$ denote the systole of S. Then $\lambda_{2g-2}(S) > \frac{1}{4} + \epsilon_0(S)$ where $\epsilon_0(S)$ can be taken any quantity smaller than*

$$min\{\frac{1}{4(g-1)}, \frac{1}{4}((\frac{\cosh\rho_0}{\sinh\rho_0})^2 - 1)\}$$

where $2s(S)\sinh\rho_0 = |S|$.

Recall that any λ_i, in particular λ_{2g-2}, is a continuous function on $\mathcal{M}_g$ (see Chapter 5). Recall that the set $\mathcal{I}_\epsilon = \{S \in \mathcal{M}_g : s(S) \geq \epsilon\}$ is compact ([Bu, p. 163]). By [O-R] $\lambda_{2g-2}(S) > 1/4$ for all $S \in \mathcal{M}_g$. Hence there exists a non-zero constant $\eta(\epsilon)$ such that $\lambda_{2g-2}(S) > 1/4 + \eta(\epsilon)$ for all $S \in \mathcal{I}_\epsilon$. This proves the Theorem with $\epsilon_0(S) = \eta(s(S))$. The content of Theorem 2.1.4 is to make this constant explicit in terms of the geometry of S i.e. one can take $\epsilon_0(S)$ any quantity below

$$\min\{\frac{1}{4(g-1)}, \frac{1}{4}((\frac{\cosh\rho_0}{\sinh\rho_0})^2 - 1)\}$$

where $2s(S)\sinh\rho_0 = |S|$.

We now briefly sketch the proof of the above theorem. Let $\mathcal{E}_\lambda$ denote the λ-eigenspace of the Laplacian on S. For $\epsilon > 0$, let $\mathcal{E}^{\frac{1}{4}+\epsilon}$ be the direct sum of eigenspaces $\mathcal{E}_\lambda$ with $\lambda \leq \frac{1}{4} + \epsilon$. For $f \neq 0 \in \mathcal{E}^{\frac{1}{4}+\epsilon}$, consider the *nodal set* $\mathcal{Z}(f)$ (see Chapter 1). It is proved in [O-R], using the analyticity of eigenfunctions on $\mathbb{H}$, that $\mathcal{Z}(f)$ is the union of a finite graph and a discrete set. Let $\mathcal{G}(f)$ be the subgraph of $\mathcal{Z}(f)$ obtained by suppressing those connected components which are homotopic to a point on S (equivalently, those which are contained in a topological disc). Due to this modification, each component of $S \setminus \mathcal{G}(f)$ is incompressible (see Chapter 1 for the definition). One of the main observation in [O-R] was that for any $f \neq 0 \in \mathcal{E}^{\frac{1}{4}}$, the Euler characteristic of at least one component of $S \setminus \mathcal{G}(f)$ is strictly negative. For $\epsilon > 0$ there is no reason, in general, to believe such a result for $f \neq 0 \in \mathcal{E}^{\frac{1}{4}+\epsilon}$. However we will prove the following lemma.

Lemma 2.1.5. *Let S be a closed hyperbolic surface of genus g. Then there exists an explicit constant $\epsilon_0(S) > 0$ depending only on the genus g and the systole of S, such that for any $f \neq 0$ $\in \mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$, the Euler characteristic of at least one component of $S \setminus \mathcal{G}(f)$ is strictly negative.*

The above Lemma will be deduced from the following:

Proposition 2.1.6. *Let S be a closed hyperbolic surface of genus g. Let $\Omega \subset S$ be a surface with smooth boundary which is homeomorphic either to a disc or to an annulus. Then there exists a constant $\epsilon(\Omega) > 0$ depending on the length l_Ω of the geodesic in S homotopic to a generator of $\pi_1(\Omega)$ and the area of Ω such that the first Dirichlet eigenvalue of Ω satisfies: $\lambda_0(\Omega) > 1/4+\epsilon(\Omega)$. Furthermore there exists an explicit constant $\epsilon_0(S) > 0$ depending only on the systole of S such that $\epsilon(\Omega) > \epsilon_0(S)$.*

Notation 2.1.7. *For any surface $\Omega \subseteq S$ with smooth boundary, $|\Omega|$ denotes the area of Ω for the area measure on S and $L(\partial\Omega)$ denote the length of the boundary of Ω.*

We shall see in the proof that $\epsilon(\Omega)$ is a strictly decreasing function of $|\Omega|$ when l_Ω is kept fixed and a strictly increasing function of l_Ω when $|\Omega|$ is kept fixed. The statement in the proposition then follows from the observation that both the parameters i.e. $|\Omega|$ and l_Ω are bounded: the first one being bounded above by $4\pi(g-1)$ and the last one being bounded below by $s(S)$.

In §3 we study the behavior of λ_i as a function on the *moduli space* $\mathcal{M}_g$. We recall that the moduli space $\mathcal{M}_g$ is the space of all closed hyperbolic surfaces of genus g up to isometry. We focus our interest on the first $2g-2$ non-zero eigenvalues. Theorem 2.1.4 (or even a continuity argument on $\mathcal{M}_g$) implies one direction of the following

Claim 2.1.8. *For a family S_n of compact hyperbolic surfaces in $\mathcal{M}_g$, $\lambda_{2g-2}(S_n)$ tends to $1/4$ if and only if the systole $s(S_n)$ tends to zero.*

The other direction follows from a construction due to P. Buser [Bu].

The above proposition can be compared with the following result of Schoen, Wolpert and Yau [S-W-Y]. Let M be a closed oriented surface of genus g with a metric of (possibly variable) Gaussian curvature K. For an integer $n \geq 1$ consider the family $\tilde{C}_n$ of curves on M which are disjoint union of simple closed geodesics and which divide M into $n+1$ components (necessarily $n \leq 2g-3$). Define a number l_n by

$$l_n = min\{L(C) : C \in \tilde{C}_n\}.$$

where $L(C)$ denotes the length of C. Then

Theorem (Schoen-Wolpert-Yau). *Suppose for some constant $k > 0$ we have $-1 \leq K \leq -k$. Then there exist positive constants α_1, α_2 depending only on g such that for $1 \leq n \leq 2g-3$, we have $\alpha_1 k^{3/2} l_n \leq \lambda_n \leq \alpha_2 l_n$ and $\alpha_1 k \leq \lambda_{2g-2} \leq \alpha_2$.*

Recall that the Bers constant β [B] which depends only on g has the property that $l_{2g-2} < \beta$. So this theorem implies that λ_{2g-2} is bounded above by a constant depending only on g. Observe also that the Buser's construction ([Bu, Theorem 8.1.3]) leads to the same conclusion. Namely by Buser's construction for any $\delta > 0$ there exists a constant $\epsilon > 0$ such that $\lambda_{2g-2} < \frac{1}{4} + \delta$ for any $S \in \mathcal{M}_g$ with $s(S) < \epsilon$. Since λ_{2g-2} is a continuous function on $\mathcal{M}_g$ and $\mathcal{I}_\epsilon = \{S \in \mathcal{M}_g : s(S) \geq \epsilon\}$ is compact the existence of an upper bound is clear. In this context we would like to mention a paper due to Dodziuk, Pignataro, Randol and Sullivan [D-P-R-S] where the authors obtained a result similar to the one of [S-W-Y] in the context of possibly non-compact hyperbolic surfaces.

After proving Claim 2.1.8, we focus on the behavior of $\lambda_i(S)$, for $1 \leq i \leq 2g-3$, as $s(S)$ tends to zero. More precisely, let $\overline{\mathcal{M}_g}$ denote the compactification of $\mathcal{M}_g$ obtained by adding the moduli spaces of (not necessarily connected) non-compact finite area hyperbolic surfaces with area equal to $4\pi(g-1)$. Let $\partial\mathcal{M}_g = \overline{\mathcal{M}_g} \setminus \mathcal{M}_g$ be the corresponding boundary of $\mathcal{M}_g$. We study the behavior of $\lambda_i(S_n)$ when $S_n \in \mathcal{M}_g$ tends to a point in $\partial\mathcal{M}_g$. By the above theorem of Schoen, Wolpert and Yau and the discussion after, it is clear that λ_i is bounded over $\mathcal{M}_g$.

Indeed the method using Buser's construction works for any i, showing that λ_i is bounded by a constant depending only on g and i. So for any i we can consider the set

$$V_i = \{\lim_{n\to\infty} \lambda_i(S_n) : (S_n) \text{ is a sequence in } \mathcal{M}_g \text{ converging to a point in } \partial\mathcal{M}_g$$

$$\text{such that } \lim_{n\to\infty} \lambda_i(S_n) \;\text{ exists}\}.$$

With this notation, the above claim says that $V_{2g-2} = \{\frac{1}{4}\}$. We next prove

Claim 2.1.9. *For any $1 \leq i \leq 2g-3$, there exists a $\Lambda_i(g) \in (0, \frac{1}{4}]$ such that V_i contains the interval $[0, \Lambda_i(g)]$.*

We shall use a result of Courtois and Colbois [C-C, Theorem 0.1] to prove this claim.

In §4 we consider finite area hyperbolic surfaces of type (g, n). The Laplace spectrum of these surfaces consists of two parts: *the continuous part* and *the discrete part.* The continuous part covers the interval $[\frac{1}{4}, \infty)$ and is spanned by the *Eisenstein series* with multiplicity n. The discrete spectrum consists of eigenvalues. They are distinguished into two parts: *the residual spectrum* and *the cuspidal spectrum.* We shall consider the cuspidal spectrum only. An eigenpair (λ, f) is called *cuspidal* if f tends to zero at each cusp. In this case λ and f are respectively called a *cuspidal eigenvalue* and a *cuspidal eigenfuction.* These eigenvalues with multiplicity are arranged by increasing order and we denote $\lambda_n^c(S)$ the n-th cuspidal eigenvalue of S.

Denote by $\mathcal{T}_{g,n}$ the *Teichmüller space* of all marked hyperbolic surfaces of type (g, n). For any choice of pair of pants decomposition of such a hyperbolic surface S one can define a system of coordinates on $\mathcal{T}_{g,n}$, the *Fenchel-Nielsen* coordinates which consists, for each curve in the pants decomposition, of the length of that curve and a *twist* parameter along that curve ([Bu, Chapter 6]). Now we consider the set ${\mathcal{T}^0}_{g,n}$ of all hyperbolic surfaces in $\mathcal{T}_{g,n}$ for which all twist parameters are equal to zero. Each surface in ${\mathcal{T}^0}_{g,n}$ carries an involution ι which when restricted to each pair of pants is the orientation reversing involution that fixes the boundary components. This involution induces an involution on each eigenspace of the Laplacian. The eigenfunctions corresponding to the eigenvalue -1 are called *antisymmetric* and the corresponding eigenvalue is called an *antisymmetric eigenvalue.* We denote the i-th antisymmetric cuspidal eigenvalue of S by ${\lambda^{o,c}}_i(S)$.

Theorem 2.1.10. *For every surface $S \in {\mathcal{T}^0}_{g,n}$ there exists an explicit constant $\epsilon_0(S) > 0$, depending only on the systole of the surface S, such that ${\lambda^{o,c}}_g(S) > 1/4 + \epsilon_0(S)$.*

Indeed, the constant $\epsilon_0(S)$ can be taken equal to any number below

$$\min\{\frac{1}{2(2g-2+n)}, \frac{1}{4}((\frac{\cosh\rho_0}{\sinh\rho_0})^2 - 1)\}$$

where $2s(S)\sinh\rho_0 = |S|$.

2.2 Proof of Proposition 2.1.6

The proof depends mainly on two geometric inequalities: the Faber-Krahn isoperimetric inequality and the Cheeger's inequality. Suppose first that $\Omega \subseteq S$ is a disc or more generally a domain such that $\pi_1(\Omega)$ maps to zero in $\pi_1(S)$. Then choose an isometric lift of Ω to $\mathbb{H}$, still denoted by Ω.

Let $B(t)$ be the geodesic disc in $\mathbb{H}$ with radius t. The geodesic disc with same area as Ω has radius $t(\Omega) = 2\sinh^{-1}(\frac{|\Omega|}{4\pi})$. By the Faber-Krahn inequality (see Chapter 1) $\lambda_0(B(t(\Omega))) \leqslant \lambda_0(\Omega)$.

Since Ω is contained in S whose area equals $2\pi(2g-2)$, by Gauss-Bonnet theorem, $|\Omega| < 2\pi(2g-2)$. Therefore, $B(t(\Omega))$ is contained in the disc with radius $t_0 = 2\sinh^{-1}(g-1)$. Recall that for two subsurfaces D_1 and D_2 in $\mathbb{H}$ with compact closure, $\lambda_0(D_1) > \lambda_0(D_2)$ when $D_1 \subsetneq D_2$. Thus $\lambda_0(B(t))$ is a strictly decreasing function of t. Hence $\lambda_0(B(t(\Omega))) > \lambda_0(B(t_0))$. Now by Theorem 5 in [Cha], we have

$$\lambda_0(B(t)) > \lim_{s\to\infty} \lambda_0(B(s)) = \frac{1}{4}.$$

Hence we finally have a strictly positive $\epsilon_1(|\Omega|)$ which depends only on the area $|\Omega|$ of Ω such that $\lambda_0(B(t(\Omega))) = \frac{1}{4} + \epsilon_1(|\Omega|)$. Since $\lambda_0(B(t))$ is a strictly decreasing function of t, $\epsilon_1(|\Omega|)$ is a strictly decreasing function of $|\Omega|$ which is bounded below by the constant $\epsilon_1(S) = \lambda_0(B(t_0)) - \frac{1}{4}$.

Suppose now that Ω is an annulus and that the image of $\pi_1(\Omega)$ in $\pi_1(S)$ is a non-trivial cyclic subgroup $\langle\tau\rangle$. Let $\mathbb{T}$ denote the *cylinder* $\mathbb{H}/\langle\tau\rangle$. Let γ denote the core geodesic of $\mathbb{T}$ and l the length of γ. Then l is the length of the shortest geodesic of S homotopic to a generator of $\pi_1(\Omega)$. Consider an isometric lift of the annulus Ω to $\mathbb{H}/\langle\tau\rangle$, still denoted by Ω. We need to prove that $\lambda_0(\Omega) > \frac{1}{4} + \epsilon_0(S)$ where $\epsilon_0(S)$ depends only on l and $|\Omega|$. We will use Cheeger's inequality (see Chapter 1) in the following form:

Cheeger inequality

Let $\Omega \subsetneq \mathbb{T}$ be a submanifold with piecewise smooth boundary. Let $h(\Omega)$ be the Cheeger constant of Ω. Then

$$\lambda_0(\Omega) \geqslant \frac{h^2(\Omega)}{4}.$$

Recall that the *Cheeger constant* of Ω is equal to $\inf\{\frac{L(\partial V)}{|V|}\}$ where V ranges over all compact submanifolds of Ω with smooth boundary.

The proof of Proposition 2.1.6 in the case of an annulus follows from Cheeger inequality and the next

Lemma 2.2.1. *Let $\Omega \subsetneq \mathbb{T}$ be a submanifold with piecewise smooth boundary and $h(\Omega)$ be the Cheeger constant of Ω. Then:*

$$h(\Omega) > 1 + \epsilon_2(|\Omega|, l),$$

for some constant $\epsilon_2(|\Omega|, l) > 0$, depending only on the area of Ω and the length l of the core geodesic of $\mathbb{T}$.

Proof. First we observe that the Cheeger constant is bounded below by the quantity $\inf\{\frac{L(\partial V)}{|V|}\}$ where V ranges over connected submanifolds of Ω. Secondly, this infimum is the same when V ranges over all discs or *essential* annuli contained in Ω. Recall that an annulus is essential when it is not homotopically trivial in $\mathbb{T}$. This is because any connected, compact submanifold $V \subseteq \Omega$ is diffeomorphic either to a disc with some discs removed or to an essential annulus with some discs removed. In both cases, taking the union of V with those removed discs, one obtains a submanifold V' which is either a disc or an essential annulus which satisfies: $L(\partial V') \leq L(\partial V)$ and $|V' \geq |V|$. Therefore $\frac{L(\partial V')}{|V'|} < \frac{L(\partial V)}{|V|}$.

Suppose now that $V \subseteq \Omega$ is diffeomorphic to a disc. By the isoperimetric inequality ([B-Z, p. 11]), one has

$$(\frac{L(\partial V)}{|V|})^2 \geq 1 + \frac{4\pi}{|V|}.$$

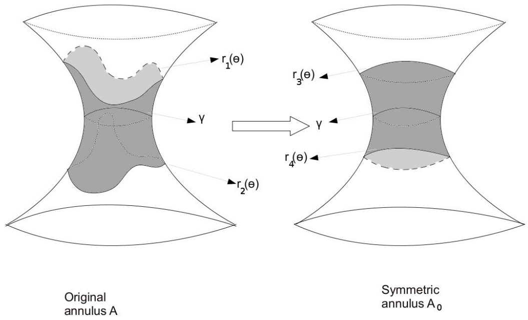
Therefore if $V \subseteq \Omega$ then $(\frac{L(\partial V)}{|V|})^2 \geq 1 + \frac{4\pi}{|\Omega|}$.

Since $|V| < 2\pi(2g-2)$, we get $(\frac{L(\partial V)}{|V|})^2 > 1 + \frac{1}{g-1}$.

Now we suppose that $V \subseteq \Omega$ is an essential annulus. In order to prove the claim in this case we will need the following notion of *symmetrization*, which is close to the notion of *Steiner symmetrization* ([He, p. 18]).

Definition 2.2.2. *Let $V \subseteq \mathbb{T}$ be an essential annulus. The symmetrization of V is the annulus $V_0 \subseteq \mathbb{T}$ symmetric with respect to γ with constant width and which has the same area V.*

Recall that the *Fermi coordinates* on $\mathbb{T}$ assign to each point p the pair $(r, s) \in \mathbb{R} \times \{\gamma\}$, where r is the signed distance of p from γ and s is the projection of p on γ. For simplicity of computation we parametrize the geodesic γ by arc-length. The resulting coordinates provide a diffeomorphism between $\mathbb{T}$ and $\mathbb{R} \times \mathbb{R}/l\mathbb{Z}$. The hyperbolic metric in these coordinates equals $dr^2 + \cosh r^2 ds^2$.



Lemma 2.2.3. *Let $V \subseteq \mathbb{T}$ be an essential annulus with piecewise smooth boundary and V_0 be the symmetrization of V. Then $L(\partial V) \geq L(\partial V_0)$.*

Proof. First we consider the case when each component of ∂V is a graph over γ. By that we mean that there exist two functions r_1 and r_2: $[0, l] \to \mathbb{R}$ such that r_i is a piecewise smooth map (there is a partition $0 = s_1 < s_2 < ... < s_m = l$ such that each restriction $r_i|[s_j, s_{j+1}]$ is smooth) with $r_i(0) = r_i(l)$ and the components of ∂V are parametrized in Fermi coordinates as: $\{(s, r_i(s)), s \in [0, l]\}$ for $i = 1, 2$. Then the components of the symmetrization V_0 of V are the graphs of the constant functions $r_3 = \rho$ and $r_4 = -\rho$ with $\rho = \sinh^{-1}(\frac{|V|}{2l})$. Up to exchanging r_1 and r_2, we may suppose that $r_1(s) > r_2(s)$ for all $0 \leq s \leq l$. Then we calculate the areas of V and V_0:

$$|V| = \int_0^l \int_{r_1(s)}^{r_2(s)} \cosh r dr ds = \int_0^l \{\sinh r_2(s) - \sinh r_1(s)\} ds$$

and

$$|V_0| = \int_0^l \int_{-\rho}^{\rho} \cosh r dr ds = \int_0^l 2. \sinh \rho ds = 2l \sinh \rho.$$

The length of ∂V_0 is

$$L(\partial V_0) = 2l \cosh \rho$$

and the length of ∂V satisfies

$$L(\partial V) = \int_0^l \{\dot{r_1}(s)^2 + 1\}^{1/2} \cosh r_1(s) ds + \int_0^l \{\dot{r_2}(s)^2 + 1\}^{1/2} \cosh r_2(s) ds$$

$$\geq \int_0^l \{\cosh r_1(s) + \cosh r_2(s)\} ds.$$

Call L_0 the constant equal to the last expression. Observe that $L(\partial V) = L_0$ if and only if $\dot{r_1}(\theta) = 0 = \dot{r_2}(\theta)$. This implies that r_1 and r_2 are constants.

One has:

$$L(\partial V)^2 - |V|^2 \geq (L_0 + |V|)(L_0 - |V|).$$

Now,

$$L_0 + |V| = \int_0^l ((\cosh r_2(s) + \sinh r_2(s)) + (\cosh r_1(s) - \sinh r_1(s))) ds$$

$$= \int_0^l (\exp(r_2(s)) + \exp(-r_1(s))) ds$$

and similarly

$$L_0 - |V| = \int_0^l (\exp(-r_2(s)) + \exp(r_1(s))) ds.$$

Thus we have

$(L_0 + |V|)(L_0 - |V|)$

$$= \left(\int_0^l (\exp\left(r_2(s)\right) + \exp\left(-r_1(s)\right)) ds\right)\left(\int_0^l (\exp\left(-r_2(s)\right) + \exp\left(r_1(s)\right)) ds\right)$$

$$\geq (\int_0^l (\exp\left(r_2(s)\right) + \exp\left(-r_1(s)\right))^{\frac{1}{2}} (\exp\left(-r_2(s)\right) + \exp\left(r_1(s)\right))^{\frac{1}{2}} ds)^2$$

(by Hölder's inequality)

$$= (\int_0^l (2 + 2\cosh\left(r_1(s) + r_2(s)\right))^{\frac{1}{2}} ds)^2.$$

Since $\cosh x \geq 1\ \forall x$, we get $(L_0 + |V|)(L_0 - |V|) \geq 4l^2 = L(\partial V_0)^2 - A(V_0)^2$. Equality holds if and only if r_1, r_2 are independent of s and if $r_1 = -r_2$.
Since by construction $|V| = |V_0|$, the lemma is proven when V is an annulus whose boundary components are graphs over γ.

Now we consider the case of an arbitrary annulus with piecewise smooth boundary. By approximation, it suffices to prove Lemma 2.2.3 for those V which satisfy the following property: there exists a partition of γ: $0 = s_1 < s_2 < ... < s_k = l = 0$ such that over each interval $[s_i, s_{i+1}]$, ∂V is the union of graphs of finitely many functions. We consider now such an annulus. We consider the strip over $[s_i, s_{i+1}]$ in $\mathbb{T}$ which is diffeomorphic to $[s_i, s_{i+1}]\times\mathbb{R}$ in Fermi coordinates. Denote by V^i the intersection of V with this strip. Let for $1 \leq i \leq k$, we denote by $f_j, j = 0, 1, 2..., l(i)$ the boundary curves of V^i i.e. in Fermi coordinates the components of ∂V^i are parametrized as $\{(s, f_j(s)) : s \in [s_i, s_{i+1}]\}$ for $j = 0, 1, 2..., l(i)$ and for any $s \in [s_i, s_{i+1}]$, $r(f_0(s)) > r(f_1(s)) > ... > r(f_{l(i)}(s))$. Now we calculate the area of V^i

$$|V^i| = \sum_{j=l(i)-1, l(i)-3,...,1} \int_{s_i}^{s_{i+1}} \int_{f_j(s)}^{f_{j+1}(s)} \cosh r dr ds = \sum_{j=1}^{l(i)} \int_{s_i}^{s_{i+1}} (-1)^{j+1} \sinh f_j(s) ds.$$

The length of ∂V^i is given by

$$L(\partial V^i) = \sum_{j=1}^{l(i)} \int_{s_i}^{s_{i+1}} \{\dot{f}_j(s)^2 + 1\}^{1/2} \cosh f_j(s) ds \geq \int_{s_i}^{s_{i+1}} \sum_{j=1}^{l(i)} \cosh f_j(s).$$

Call $L_0(i)$ the constant equal to the last expression and calculate

$$L(\partial V)^2 - |V|^2 = (\sum_i L(\partial V^i))^2 - (\sum_i |V^i|)^2 \geq (\sum_i L_0(i))^2 - (\sum_i |V^i|)^2$$

$$= (\sum_i \int_{s_i}^{s_{i+1}} (\sum_{j=1}^{l(i)} \exp[(-1)^{j+1} f_j(s)] ds) \times (\sum_i \int_{s_i}^{s_{i+1}} (\sum_{j=1}^{l(i)} \exp[(-1)^{j} f_j(s)] ds)$$

$$\geq (\sum_i \int_{s_i}^{s_{i+1}} (\exp[(-1)^{0+1} f_0(s)] + \exp[(-1)^{1+1} f_1(s)]) ds)$$

$$\times (\sum_i \int_{s_i}^{s_{i+1}} (\exp[(-1)^{0} f_0(s)] + \exp[(-1)^{1} f_1(s)]) ds)$$

$$\geq (\int_0^l (2 + 2\cosh(f_1(s) - f_0(s))^{\frac{1}{2}} ds)^2$$

$$\text{(by Hölder's inequality)}$$

$$\geq 4l^2 = L(\partial V_0)^2 - |V_0|^2.$$

Hence using the same argument as before we finally prove Lemma 2.2.3. □

So now we have $(\dfrac{L(\partial V)}{|V|}) \geq (\dfrac{L(\partial V_0)}{|V_0|}) = \dfrac{\cosh \rho}{\sinh \rho}$ where $|V| = 2l \sinh \rho$. Thus we conclude the proof of Lemma 2.2.1 by taking

$$\epsilon_2(\Omega, l) = \frac{1}{2}\min\{\frac{\cosh \theta}{\sinh \theta} - 1, (1 + \frac{4\pi}{|\Omega|})^{\frac{1}{2}} - 1\}$$

where $|\Omega| = 2l \sinh\theta$. $\square$

Since $\frac{\cosh\rho}{\sinh\rho}$ is a strictly decreasing function of ρ we have

$$(\frac{L(\partial V)}{|V|}) \geq \frac{\cosh\rho_1}{\sinh\rho_1} > \frac{\cosh\rho_0}{\sinh\rho_0}$$

where $2l\sinh\rho_1 = |\Omega|$ and $2s(S)\sinh\rho_0 = |S| = 4\pi(g-1)$ (since $V \subseteq \Omega \subsetneq S$). To conclude the proof of Proposition 2.1.6 we take

$$\epsilon_0(S) = \frac{1}{2}\min\{\epsilon_1(S), \frac{1}{4(g-1)}, \frac{1}{4}((\frac{\cosh\rho_0}{\sinh\rho_0})^2 - 1)\}.\square$$

Remark 2.2.4. *From the expression of $\epsilon_0(S)$ we observe that if (S_n) be a sequence in $\mathcal{M}_g$, then $\epsilon_0(S_n)$ tends to zero only if $s(S_n)$ tends to zero. The computations in the proposition also show that for any $\Omega \subseteq S$ diffeomorphic to a disc or to an annulus one has*

$$\lambda_0(\Omega) \geq \frac{1}{4} + 2\epsilon_0(S).$$

2.2.5 Proof of Theorem 2.1.4

The proof at this point follows the same lines as that of [O-R, Théorèm 0.2] and we refer [O-R] for the details. We take $\epsilon_0(S)$ as in Proposition 2.1.6. Consider the space $\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$. Recall that $\mathcal{E}^{\lambda}$ is the direct sum of the eigenspaces of the Laplacian with eigenvalues less than or equal to λ. Let $f \neq 0 \in \mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$. The *nodal set* $\mathcal{Z}(f)$ of f is defined as $f^{-1}(0)$. Recall that $\mathcal{G}(f)$ is the subgraph of $\mathcal{Z}(f)$ obtained by suppressing those connected components which are zero homotopic on S. Each component of $S \setminus \mathcal{G}(f)$ is an open surface, may be equal to S when $\mathcal{G}(f)$ is empty. The sign of f on a component of $S \setminus \mathcal{G}(f)$ can be defined as follows. There is a finite collection of disjoint closed topological discs (D_j) with $\partial D_j \bigcap \mathcal{Z}(f) = \phi$ such that each component of $\mathcal{Z}(f)$ which is zero homotopic is contained in one of the D_j's. Therefore each component of $S \setminus \mathcal{G}(f)$ is a union of a component of $S \setminus \mathcal{Z}(f)$ with a finite number of those D_j's. Define the sign of f on such a component to be the one of f on the corresponding component if $S \setminus \mathcal{Z}(f)$. Now we denote the union of all components with positive (resp. negative) sign as $C^+(f)$ (resp. $C^-(f)$). As a consequence of the construction, the surfaces $C^+(f)$ and $C^-(f)$ are incompressible. As recalled earlier, an open subset of a surface S is called incompressible if the fundamental group of any of its connected components maps injectively into $\pi_1(S)$. The union of the connected components of $C^+(f)$ (resp. $C^-(f)$) which are neither discs nor rings is denoted by $S^+(f)$ (resp. $S^-(f)$). The surfaces $S^{\pm}(f)$ may be empty or disconnected but by construction when they are nonempty, they are incompressible.

Denote the Euler characteristic of $S^+(f)$ (resp. $S^-(f)$) by $\chi^+(f)$ (resp. $\chi^-(f)$). (we use the convention that the Euler characteristic of the empty set is zero). The incompressibility property of $S^+(f)$ and $S^-(f)$ gives that $\chi^+(f) + \chi^-(f)$ is greater than $\chi(S)$. By definition, we have $\chi^{\pm}(f) \leq 0$ with equality only if $S^{\pm}(f)$ is empty.

Lemma 2.1.5 *The Euler characteristic of at least one component of $S \setminus \mathcal{G}(f)$ is negative.*

Proof. Let us suppose by contradiction that for some $f \neq 0 \in \mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$, each component S_i, $1 \leq i \leq m$ of $S \setminus \mathcal{G}(f)$ has non-negative Euler characteristic. So, each such component is homeomorphic either to an open disc or to an open annulus. Since $f \in \mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$ the Rayleigh quotient of f, $R(f)$ is $\leq \frac{1}{4} + \epsilon_0(S)$. Therefore, since $\mathcal{G}(f)$ has measure zero, for at least one component, say S_1, one has

$$R(f|S_1) = \frac{\int_{S_1} \| \nabla f \|^2}{\int_{S_1} f^2} \leq \frac{1}{4} + \epsilon_0(S).$$

Now we shall calculate the Rayleigh quotient $R(f|S_1)$ and show that our choice of $\epsilon_0(S)$ leads to a contradiction.

Let us assume that S_1 is homeomorphic to an open disc. The case when S_1 is an annulus can be dealt with similarly. Since f is smooth and $f|\partial S_1 = 0$ we can choose, by Sard's theorem, a sequence (ϵ_n) of regular values of f converging to 0. Then the level set $\{x \in S_1 : f(x) = \epsilon_n\}$ is a smooth submanifold of S for n large enough. Furthermore one of the components of this level set confines a domain $D_n \subsetneq S_1$ homeomorphic to a closed disc with smooth boundary such that $S_1 \setminus D_n$ has arbitrarily small area. Now we consider the Rayleigh quotient, $R(f_n|D_n)$ of the function $f_n = f - \epsilon_n$ restricted to the region D_n. This function vanishes on ∂D_n. As ϵ_n converges to 0, $R(f_n|D_n)$ converges to $R(f|S_1)$. Thus for any $\delta > 0$, in particular for $\dfrac{\epsilon_0(S)}{2}$, we can find ϵ_n small enough such that $R(f_n|D_n) \leq \dfrac{1}{4} + \epsilon_0(S) + \dfrac{\epsilon_0(S)}{2} < \dfrac{1}{4} + 2\epsilon_0(S)$. Now since D_n is a closed disc with smooth boundary which is contained in $S_1 \subseteq S$, it follows from the Rayleigh quotient characterization of the first Dirichlet eigenvalue of D_n that $R(f_n|D_n) \geq \lambda_0(D_n)$. By Remark 2.2.4 we have $\lambda_0(D_n) \geq \dfrac{1}{4} + 2\epsilon_0(S)$. This is a contradiction when n is sufficiently large. □

So some component of $S \setminus \mathcal{G}(f)$ has negative Euler characteristic. This component is a component of $S^{\pm}(f)$. Thus we obtain:

$$\chi^+(f) + \chi^-(f) < 0.$$

Now we start with some definitions and complete the proof.

Definition 2.2.6. *According to the sign of f on S_i, we denote this component as ${S_i}^+(f)$ or ${S_i}^-(f)$. For each such surface with negative Euler characteristic, we consider a compact core, i.e. a compact surface ${K_i}^{\pm}(f) \subset {S_i}^{\pm}(f)$ such that the inclusion is a homotopy equivalence. We then define the surface $\Sigma^+(f)$ (resp. $\Sigma^-(f)$) as the union of the compact cores ${K_i}^+(f)$ (resp. ${K_i}^-(f)$) and of those components (if any) of the complement $S \setminus \bigcup {K_i}^+(f)$ (resp. $S \setminus \bigcup {K_i}^-(f)$), which are annuli. Therefore, $\Sigma^+(f)$ (resp. $\Sigma^-(f)$) is obtained from $\bigcup {K_i}^+(f)$ (resp. $\bigcup {K_i}^-(f)$), by adding (if any) the annuli between the components of $\bigcup {K_i}^+(f)$ (resp. $\bigcup {K_i}^-(f)$). We call $\Sigma(f) = \Sigma^+(f) \bigcup \Sigma^-(f)$, the characteristic surface of f, while $\Sigma^+(f)$ (resp. $\Sigma^-(f)$) is called the positive (resp. negative) characteristic surface of f. The definition of these surfaces depend uniquely on the choice of compact cores and those are well defined up to isotopy. By construction the Euler characteristic of $\Sigma^+(f)$ (resp. $\Sigma^-(f)$) is $\chi^+(f)$ (resp. $\chi^-(f)$). It is clear that $\Sigma^+(-f) = \Sigma^-(f)$ and $\Sigma^-(-f) = \Sigma^+(f)$.*

Continuation of the proof of Theorem 2.1.4.

Let m denote the dimension of the space $\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$. Theorem 2.1.4 will follow from the inequality $m \leq (2g-2)$. Let $\mathbb{S}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)})$ denote the unit sphere of $\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$ (for some arbitrary norm) and let $\mathbb{P}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)})$ be the projective space of $\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$ i.e. the quotient of $\mathbb{S}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)})$ by the involution $f \to -f$.
For each integer i with $2 - 2g \leq i \leq -1$, we denote

$$C_i = \{f \in \mathbb{S}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}) | \chi^+(f) + \chi^-(f) = i\}.$$

According to the lemma and its consequence above, $\mathbb{S}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}) = \bigcup_{2-2g}^{-1} C_i$. On the other hand, each C_i is invariant under the antipodal involution. Let P_i be the quotient of C_i under the antipodal involution. The projective space $\mathbb{P}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)})$ is the union of the sets P_i.

Lemma 2.2.7. *For any integer i, $2-2g \leq i \leq -1$, the covering map $C_i \to P_i$ is trivial.*

Proof. Let $f \in C_i$. We use the notations introduced in the definition of characteristic surface of f: ${S_i}^{\pm}(f)$ is a connected component of negative Euler characteristic of $S^{\pm}(f)$ and ${K_i}^{\pm}(f)$ is a compact core of ${S_i}^{\pm}(f)$. We may assume that the compact core has been chosen in such a way that any connected component of $Z(f)$ that is contained in some ${S_i}^{\pm}(f)$ is indeed contained in the interior of the corresponding ${K_i}^{\pm}(f)$.

For any function $g \in \mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}$ close enough to f, and for each i, ${K_i}^{\pm}(f)$ is contained in a component ${S_l}^{\pm}(g)$ of $S^{\pm}(g)$. Fix a neighborhood $V(f)$ of f in $\mathbb{S}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)})$ such that these inclusions occur on each surface ${K_i}^{\pm}(f)$.

We will show that for any $g \in C_i \cap V(f)$, the characteristic surfaces $\Sigma^+(f)$ and $\Sigma^+(g)$ (resp. $\Sigma^-(f)$ and $\Sigma^-(g)$) are isotopic. Choose the compact cores ${K_i}^{\pm}(g)$ of surfaces $S^{\pm}(g)$ so that when ${K_i}^{\pm}(f)$ is contained in ${S_j}^{\pm}(g)$, it is also contained in the interior of ${K_j}^{\pm}(g)$. Now observe that if two components of the boundaries of surfaces ${K_j}^{+}(f)$ are homotopic in S then the homotopy between them is achieved by an annulus contained in $\Sigma^+(f)$, by the definition of the characteristic surface. Since this annulus joins two curves of ${K_j}^{+}(g)$ by the definition of the characteristic surface again, it is contained in one of the connected components $\Sigma^+(g)$ too.

We deduce from this that each connected component of $\Sigma^{\pm}(f)$ is contained in a connected component of $\Sigma^{\pm}(g)$ (of the same sign). Since $\Sigma^+(f)$ and $\Sigma^-(f)$ are incompressible in S, they are incompressible in $\Sigma^+(g)$ and $\Sigma^-(g)$ respectively. In particular, their Euler characteristic satisfy

$$\chi^+(f) \leq \chi^+(g) \text{ and } \chi^-(f)) \leq \chi^-(g);$$

these inequalities can be equalities if and only if the surfaces $\Sigma^+(f)$ and $\Sigma^+(g)$ (resp. $\Sigma^+(f)$ and $\Sigma^+(g)$) are isotopic. But since $g \in C_i$, we have

$$\chi^+(f) + \chi^-(f) = i = \chi^+(g) + \chi^-(g).$$

Thus $\Sigma^+(f)$ and $\Sigma^+(g)$ are isotopic. The same holds for $\Sigma^-(f)$ and $\Sigma^-(g)$.

Since the *isotopy class of* $\Sigma^+(f)$ and *isotopy class of* $\Sigma^-(f)$ are locally constant on C_i, they are constant on each connected component of C_i. Finally we observe that the functions f and $-f$ can not be in the same connected component of C_i. This is because then $\Sigma^+(f)$ and $\Sigma^-(f)$ would be isotopic. But two disjoint and incompressible surfaces of negative Euler characteristic contained in S can not be isotopic. Thus the covering map in Lemma 2.2.7 is trivial. □

Continuation of the proof of Theorem 2.1.4

We conclude the proof of the Theorem following a method of B. Sévennec [Se]. The double covering $\mathbb{S}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}) \to \mathbb{P}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)})$ is associated to a cohomology class $\beta \in H^1(\mathbb{P}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)}), \mathbb{Z}/2\mathbb{Z})$. Each covering $C_i \to P_i$ is described by the Cech cohomology class, $\beta|_{P_i}$. Since each of this covering is trivial, we have $\beta|_{P_i} = 0$. Since $\mathbb{P}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)})$ is the union of P_i and since there are at most $2g-2$ of them, we have: $\beta^{2g-2} = 0$([Se, Lemma 8]). Since β has order m in the $\mathbb{Z}/2\mathbb{Z}$-cohomology ring of $\mathbb{P}(\mathcal{E}^{\frac{1}{4}+\epsilon_0(S)})$, we have $m \leq 2g-2$.□

2.3 Systole and the Laplace spectrum

In this section we study the eigenvalues of the Laplacian as functions on the moduli space. Recall that the moduli space $\mathcal{M}_g$ is the space of all closed hyperbolic surfaces of genus g up to isometry. $\mathcal{M}_g$ can be compactified to a space $\overline{\mathcal{M}_g}$ by adding the moduli spaces of (not necessarily connected) non-compact finite area hyperbolic surfaces with area equal to $4\pi(g-1)$. In this compactification a sequence (S_n) in $\mathcal{M}_g$, with $s(S_n) \to 0$, converges to $S_\infty \in \mathcal{M}_{g_0,n_0}$ (with $2g_0 - 2 + n_0 = 2g - 2$) if and only if for any given $\epsilon > 0$ the ϵ-thick part $({S_n}^{[\epsilon,\infty)})$ converge to ${S_\infty}^{[\epsilon,\infty)}$ in the Gromov-Hausdorff topology. Recall that the ϵ-thick part of a surface S is the subset of those points of S where the *injectivity radius* is at least ϵ. Recall also that the injectivity radius of a point $p \in S$ is the radius of the largest geodesic disc that can be embedded in S with center p.

It is a classical result that for any i, λ_i is a continuous function on $\mathcal{M}_g$ (see Chapter 5). Moreover, it is shown in [C-C] that eigenvalues less than $1/4$ are continuous up to $\partial\mathcal{M}_g$. We shall discuss results of this type in Chapter 3. In this section we focus on the following. For a fixed i we shall study the behavior of $\lambda_i(S_n)$ when $S_n \in \mathcal{M}_g$ tends to a point in $\partial\mathcal{M}_g$. Recall

$$V_i = \{\lim_{n\to\infty} \lambda_i(S_n) : (S_n) \text{ is a sequence in } \mathcal{M}_g \text{ converging to a point in } \partial\mathcal{M}_g$$

$$\text{such that the limit exists}\}.$$

In [R3] Randol showed a limiting behavior of λ_{2g-2} over some special family. Now we apply Theorem 2.1.4 to prove the following,

Claim 2.1.8 *$\lambda_{2g-2}(S_n)$ tends to $\frac{1}{4}$ if and only if $s(S_n)$ tends to zero. In particular $V_{2g-2} = \{\frac{1}{4}\}$.*

Proof. By Theorem 2.1.4, if $\lambda_{2g-2}(S_n)$ tends to $\frac{1}{4}$ then $\epsilon_0(S_n)$ tends to zero. For the other direction we use Buser's construction. By the definition of the systole, there is a closed geodesic τ on S such that the length of τ is equal to $s(S)$. Now from the *Collar Theorem* (ref. [Bu]) of L. Keen[K] (see also [R2]) and the explicit computations in [Bu, p. 219] we see that for any $\epsilon > 0$ and any $i \geq 1$ we have $\delta > 0$ such that whenever $s(S) < \delta$, we can find at least i disjoint annuli in the collar neighborhood of τ of length such that the first Dirichlet eigenvalue of each of the annuli is $\leq \frac{1}{4} + \epsilon$. The corresponding eigenfunctions are orthogonal. Hence we have $\lambda_{i-1}(S) \leq \frac{1}{4} + \epsilon$. Therefore using Theorem 2.1.4 for an $i \geq 2g-1$ we obtain the convergence $\lambda_i(S_n) \to \frac{1}{4}$.□

Now we show that such a limiting behavior is not true in general for $i \leq 2g-3$. Moreover

Claim 2.1.9 *For any $1 \leq i \leq 2g-3$, there exists $\Lambda_i(g)$, $0 < \Lambda_i(g) \leq \frac{1}{4}$ such that $V_i = [0, \Lambda_i(g)]$.*

Before starting the proof we recall the definition of Teichmüller space, $\mathcal{T}_g$. It is the space of all marked closed hyperbolic surfaces of genus g. Let $S \in \mathcal{T}_g$. Given a pair of pants decomposition of S, we have a coordinate system on $\mathcal{T}_g$, the Fenchel-Nielsen coordinates. $\mathcal{M}_g$ is the quotient of $\mathcal{T}_g$ by the action of Mod_g, the *Teichmüller modular group*. Since Mod_g acts properly discontinuously on $\mathcal{T}_g$, $\mathcal{T}_g \to \mathcal{M}_g$ is a ramified topological covering. Thus the pre-composition of this covering map with λ_i yields a map, also denoted by λ_i, from $\mathcal{T}_g$ to $\mathbb{R}$. We shall use the same notation for a point in $\mathcal{T}_g$ and its image in $\mathcal{M}_g$ too.

Proof. We shall prove the claim for $i = 1$. The proof for $1 \leq i \leq 2g-3$ is similar. We choose a pair of pants decomposition $\mathcal{P}$ of a $S \in \mathcal{T}_g$ and consider the corresponding Fenchel-Nielsen coordinates $({l^{\mathcal{P}}}_j, {\theta^{\mathcal{P}}}_j)$ on $\mathcal{T}_g$. Here ${l^{\mathcal{P}}}_j$'s denote the *length* coordinates and ${\theta^{\mathcal{P}}}_j$'s denote the *twist* coordinates (ref. [Bu]). We fix two geodesics γ and γ' among the boundary geodesics of the pants decomposition $\mathcal{P}$. Thus the length functions l_γ and $l_{\gamma'}$ respectively of γ and $\gamma^{'}$ are among ${l^{\mathcal{P}}}_j$'s. Suppose that the pants decomposition is chosen in such a way that γ is non-separating and $\gamma^{'}$ is separating.

First we prove that V_i is not empty. From a construction of P. Buser [Bu, Theorem 8.1.3] it follows that if $0 < \delta < \frac{1}{24}$ then $\lambda_{2g-3}(S) < \frac{1}{4}$ for any $S \in \mathcal{T}_g$ with ${l^{\mathcal{P}}}_j(S) < \delta$ for all j (the number $\frac{1}{24}$ has no particular significance other than ensuring this condition). We fix one such δ and consider one $M \in \mathcal{T}_g$ such that ${l^{\mathcal{P}}}_j(M) < \delta$ for all j. Now consider a sequence of surfaces $(S_n) \in \mathcal{T}_g$ such that $({l^{\mathcal{P}}}_j, {\theta^{\mathcal{P}}}_j)(S_n) = ({l^{\mathcal{P}}}_j, {\theta^{\mathcal{P}}}_j)(M)$ for all $({l^{\mathcal{P}}}_j, {\theta^{\mathcal{P}}}_j)$ except l_γ and the $l_\gamma(S_n)$ coordinate decreases to zero as n goes to infinity. Then (S_n) converges to a point $S_\infty \in \partial\mathcal{M}_g$. By our choice of δ (for M) and since the number of components of S_∞ is exactly one, it follows from [C-C, Theorem 0.1] that $0 < \lim_{n\to\infty}\lambda_1(S_n) = \lambda_1(S_\infty) = p < \frac{1}{4}$. Now consider another sequence $({S^{'}}_n)$, constructed in the same way as (S_n) except by varying the coordinate $l_{\gamma'}$ instead of l_γ. In this case the limiting surface of the sequence $({S^{'}}_n)$ has two components. So using [C-C] again $\lim_{n\to\infty}\lambda_1({S^{'}}_n) = 0$. Thus we see that 0 and $p \in V_1$, proving that V_1 is not empty.

Next we prove that whenever some $0 < c \leq \frac{1}{4}$ is in V_1, the whole interval $(0, c]$ is contained in V_1. Since c is in V_1 we have a sequence (P_n) in $\mathcal{M}_g$ such that $\lim_{n\to\infty}\lambda_1(P_n) = c$. Up to extracting a subsequence, we might assume that (P_n) converges to $P_\infty \in \partial\mathcal{M}_g$. Then P_∞ is a finite area connected (since $c > 0$) non-compact hyperbolic surface of type $(g^{'}, m)$ (where $g^{'} + \frac{m}{2} = g$). For some marking of P_n, there is a pants decomposition of S, $\gamma_1, ..., \gamma_k, ..., \gamma_{3g-3}$ such that $\gamma_1, ..., \gamma_k$ are exactly those curves on P_n whose lengths tends to zero. Consider the corresponding Fenchel-Nielsen coordinates $(l_i, \theta_i)_{i=1,2,..,3g-3}$ on $\mathcal{T}_g$. These coordinates induce coordinates on $\mathcal{T}_{g',m}$ which will be denoted by the same notation. In these coordinates we can choose representatives of P_n in $\mathcal{T}_g$ such that $({l_i}^n, {\theta_i}^n)(P_n)$ converges to $({l_i}^\infty, {\theta_i}^\infty)$ for $i > k$ and for $i \leq k$, ${l_i}^n$ converges to zero. Next, using the Buser construction ([Bu, Theorem 8.1.3]), we choose a $N_\infty \in \mathcal{T}_{g',m}$ such that $\lambda_1(N_\infty) = \epsilon < c$. Then by [C-C, Theorem 0.1] for any sequence (N_n) in $\mathcal{M}_g$ converging to N_∞ in $\partial\mathcal{M}_g$, one has $\lim_{n\to\infty}\lambda_1(N_n) = \epsilon$. In particular we consider the sequence (N_n) such that $(l_i, \theta_i)(N_n) = (l_i, \theta_i)(N_\infty)$ for $i > k$ and $(l_i, \theta_i)(N_n) = (l_i, \theta_i)(P_n)$ for $i \leq k$. Then $\lim_{n\to\infty}\lambda_1(N_n) = \epsilon$.

At this point we construct a path σ_n in $\mathcal{M}_g$ joining P_n and N_n for each n. Let us consider the path given by the coordinate axes i.e. the path first goes along the l_i axes from $l_i(P_n)$ to $l_i(N_n)$ for each $i = k+1, k+2, .., 3g-3$ in the increasing order and then the same for θ_i's. Finally for any $t \in [\epsilon, c]$ we apply the continuity property of λ_1 on $\mathcal{M}_g$ to get a surface Q_n on σ_n such that $\lambda_1(Q_n) = t$. By construction each point on σ, in particular Q_n, has $(l_i, \theta_i)(Q_n) = (l_i, \theta_i)(P_n)$ for $i \leq k$ and all other $(l_i, \theta_i)(Q_n)$ are bounded by the corresponding coordinates of P_∞ and N_∞. Hence Q_n converges to a point $Q_\infty \in \partial\mathcal{M}_g$ and since $\lambda_1(Q_n) = t$ for each n we have $\lim_{n\to\infty}\lambda_1(Q_n) = t$. Therefore V_1 contains $[\epsilon, c]$ and ϵ being arbitrary, the whole of $(0, c]$. That $\Lambda_i(g) \leq \frac{1}{4}$ follows from the last claim. □

2.4 Non-compact finite area hyperbolic surfaces.

In this section we study non-compact finite area hyperbolic surfaces. Recall that $\mathcal{T}_{g,n}$ denotes the Teichmüller space of all marked hyperbolic surfaces with finite area and of geometric type (g, n). Given any pair of pants decomposition of any $S^{'} \in \mathcal{T}_{g,n}$ one can consider the Fenchel-Nielsen coordinates on $\mathcal{T}_{g,n}$. Fix one such coordinate system on $\mathcal{T}_{g,n}$. Denote by ${\mathcal{T}_{g,n}}^0$ the set of all surfaces in $\mathcal{T}_{g,n}$ all of whose twist parameters are equal to zero. Recall that each surface in ${\mathcal{T}^0}_{g,n}$ carries an involution ι which when restricted to each pair of pants is the orientation reversing involution that fixes the boundary components. This involution induces an involution on each eigenspace of the Laplacian. The eigenfunctions corresponding to the eigenvalue -1 are called *antisymmetric* and the corresponding eigenvalue is called an *antisymmetric eigenvalue.* We denote the i-th antisymmetric cuspidal eigenvalue of S by $\lambda^{o,c}{}_i(S)$.

We observe that in Proposition 2.1.6 we have considered domains in S which are diffeomorphic either to discs or to annuli. Since S is compact, the domains have compact closures. Now for $S_0 \in \mathcal{T}_{g,n}$, we may have nodal domains whose closure is not compact. To tackle this problem we consider only those domains which are diffeomorphic either to discs or to annuli and where respective boundary curves are not homotopic to puncture. For any such disc or annulus, the Cheeger's inequality is still true (ref. [Cha]). The computations in Lemma 2.2.1 then apply. Therefore for any $\Omega \subseteq S_0$ diffeomorphic either to a disc or to an annulus whose boundary curves are not homotopic to a puncture, we have an explicit constant $\epsilon_0(S_0) > 0$ such that

$$\lambda_0(\Omega) \geq \frac{1}{4} + 2\epsilon_0(S_0) > \frac{1}{4} + \epsilon_0(S_0).$$

Theorem 2.4.1. *For any $S_0 \in \mathcal{T}_{g,n}{}^0$ there exists an explicit constant $\epsilon_0(S_0) > 0$, depending only on the systole of the surface S_0, such that $\lambda_g^{o,c}(S_0) > \frac{1}{4} + \epsilon_0(S_0)$.*

Proof. The proof proceeds along the same lines as that of Theorem 2.1.4. We choose $\epsilon_0(S_0)$ as above and consider $\mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)}$, the subspace of $C^\infty(S_0)$, spanned by the anti-symmetric cuspidal eigenfunctions with eigenvalue $\leq \frac{1}{4} + \epsilon_0(S_0)$. Then we use the same arguments as in Theorem 2.1.4 to prove that the dimension of $\mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)}$ is less than g. First for $f \neq 0 \in \mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)}$ we consider the subgraph $\mathcal{G}(f)$ of $\mathcal{Z}(f)$ obtained by suppressing those components of $\mathcal{Z}(f)$ which are bounded and homotopic to a point in S_0 (equivalently, those which are contained in a bounded disc in S_0). Next we consider the components of $S_0 \setminus \mathcal{G}(f)$ with their signs attached as defined in 2.2.5. Denote by $\mathcal{F}(\iota)$ the fixed point set of the isometry ι. The set $\mathcal{F}(\iota)$ divides S_0 into two isometric components $\mathcal{S}_1$ and $\mathcal{S}_2$. Each $\mathcal{S}_i$ is a non-compact finite area hyperbolic surface with geodesic boundary and genus 0. Each puncture of S_0 gives rise to two ideal points, one on $\partial\mathcal{S}_1$ and another on $\partial\mathcal{S}_2$.

Claim 2.4.2. *For any $f \neq 0 \in \mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)}$ each component of $S_0 \setminus \mathcal{G}(f)$ is contained in one of the $\mathcal{S}_i$'s and is incompressible there.*

Proof. By antisymmetry of f with respect to ι we have $\mathcal{F}(\iota) \subseteq \mathcal{Z}(f)$. Since each bounded component of $\mathcal{F}(\iota)$ is incompressible therefore $\mathcal{F}(\iota) \subseteq \mathcal{G}(f)$. Hence the claim follows.□

Now we can argue as in the proof of Lemma 2.1.5 to conclude that the Euler characteristic of at least one component of $S_0 \setminus \mathcal{G}(f)$ is negative. In fact using the symmetry of $\mathcal{G}(f)$ with respect to ι, the Euler Characteristic of at least one component of $\mathcal{S}_j \setminus \mathcal{G}(f)$ is negative for each $j = 1, 2$. Next we consider the unit sphere $\mathbb{S}(\mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)})$ and the projective space $\mathbb{P}(\mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)})$ over $\mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)}$. Define $\chi^+(f)$ (respectively $\chi^-(f)$) as the sum of the Euler characteristic of the components of $\mathcal{S}_1 \setminus \mathcal{G}(f)$ with positive sign (respectively negative). Consider the decomposition of $\mathbb{S}(\mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)})$ into sets

$$\mathcal{C}_i = \{f \in \mathbb{S}(\mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)}) : \chi^+(f) + \chi^-(f) = i\}$$

The arguments in Lemma 2.2.7 can be applied. Using the incompressibility of components of $\mathcal{S}_1 \setminus \mathcal{G}(f)$ the possible values of $\chi^+(f) + \chi^-(f)$ are at most $(g-1)$ (since $\chi(\mathcal{S}_i) = 1 - g$) for any $f \in \mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)}$. Exactly the same arguments as in Lemma 2.2.7 work to prove that for any integer i, the covering map $C_i \to P_i$ is trivial. We conclude that the dimension of $\mathcal{E}_o{}^{\frac{1}{4}+\epsilon_0(S_0)}$ is $\leq g$.□

Chapter 3

Behavior of eigenpairs on converging family of hyperbolic surfaces

In this chapter we study the behavior of eigenpairs on a converging sequence of hyperbolic surfaces. We shall first recall the notion of convergence of a sequence of hyperbolic surfaces in $\overline{\mathcal{M}_{g,n}}$. Then we consider a sequence (S_m) in $\mathcal{M}_{g,n}$ that converges to $S_\infty \in \overline{\mathcal{M}_{g,n}}$. We consider an eigenpair (λ_m, ϕ_m) of S_m such that $\lambda_m \to \lambda_\infty$ as $m \to \infty$. Then we recall two results, one due to G. Courtois and B. Colbois and another one due to D. Hejhal. These two results focus on the case $\lambda_\infty < \frac{1}{4}$. Then we recall two more results, one due to Lizhen Ji [Ji, Theorem 1.5] and another one due to Scott Wolpert [Wo, Theorem 3.4] that concerns the same question without the particular restriction $\lambda_\infty < \frac{1}{4}$. However, Wolpert's result has the assumption $\lambda_\infty > \frac{1}{4}$. In the last section we focus on the case of small cuspidal eigenpairs. Motivated mainly by the last two results mentioned above we prove that the same conclusions as in these two theorems hold in this particular case.

3.1 Convergence of hyperbolic surfaces

Recall that $\mathcal{M}_{g,n}$ is the *moduli space* of hyperbolic surfaces of genus g with n puncture up to the equivalent relation of isometry. We begin with a noncompact, finite area hyperbolic surface S of *geometric type* (g, n) i.e. $S \in \mathcal{M}_{g,n}$. The Laplace spectrum of such a surface is composed of two parts: *the continuous part* and *the discrete part* [I]. The continuous part covers the interval $[\frac{1}{4}, \infty)$ and is spanned by the *Eisenstein series* with multiplicity n. Eisenstein series E's are not eigenfunctions although they satisfy

$$\Delta E(., s) = s(1-s)E(., s),$$

because they are not in L^2. For this reason, they are *generalized eigenfunctions.* The discrete spectrum consists of eigenvalues. They are distinguished into two parts: *the residual spectrum* and *the cuspidal spectrum.* An eigenpair (λ, f) is called *residual* if f is a residue of meromorphic continuations of Eisenstein series. Such λ and f are then called a *residual eigenvalue* and a *residual eigenfuction.* The residual spectrum is a finite set contained in $[0, \frac{1}{4})$. On the other hand, an eigenpair (λ, f) is called *cuspidal* if f tends to zero at each cusp. In this case λ and f are respectively called a *cuspidal eigenvalue* and a *cuspidal eigenfuction.* These eigenvalues with multiplicity are arranged by increasing order and we denote $\lambda_n^c(S)$ the n-th cuspidal eigenvalue of S. For an arbitrary Fuchsian group Γ, it is not known whether the cardinality of the set of cuspidal eigenvalues of $\mathbb{H}/\Gamma$ is infinite. However a famous theorem of A. Selberg says that it is the case when Γ is arithmetic. Any cuspidal eigenpair (λ, f) with $\lambda \leq \frac{1}{4}$ is called a *small cuspidal eigenpair* and in that case, λ and f are respectively called a *small cuspidal eigenvalue* and a *small cuspidal eigenfunction*.

The set $\mathcal{M}_{g,n}$ carries a topology in which two surfaces $\mathbb{H}/\Gamma$ and $\mathbb{H}/\Gamma^{'}$ are close when the groups Γ and $\Gamma^{'}$ can be conjugated inside $\mathrm{PSL}(2,\mathbb{R})$ so that they have generators which are close. With this topology $\mathcal{M}_{g,n}$ is not compact. However it can be compactified by adjoining $\cup_i \mathcal{M}_{g_i,n_i}$'s for each $(g_1, n_1), ..., (g_k, n_k)$ with $2\sum_i^k (g_i - 2) + \sum_i^k n_i = 2g - 2 + n$. In this compactification a sequence $(S_n) \in \mathcal{M}_{g,n}$ converges to $S_\infty \in \overline{\mathcal{M}_{g,n}}$ if and only if for any given $\epsilon > 0$ the ϵ-thick part $(S_n^{[\epsilon,\infty)})$ converges to $S_\infty^{[\epsilon,\infty)}$ in the Gromov-Hausdorff topology. Recall that the ϵ-thick part of a surface S is the subset of those points of S where the *injectivity radius* is at least ϵ. Recall also that the injectivity radius of a point $p \in S$ is the radius of the largest geodesic disc that can be embedded in S with center p.

Now we recall another point of view of this convergence which is convenient for our purpose.

Let $S_m = \mathbb{H}/\Gamma_m$ and let $0 < c_0 < \frac{1}{2}$ be a fixed constant. Let $x_m \in S_m^{[c_0,\infty)}$. Up to a conjugation of Γ_m in $\mathrm{PSL}(2,\mathbb{R})$, one may assume that the point $i \in \mathbb{H}$ is mapped to x_m under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$. Then up to extracting a subsequence we may suppose that Γ_m converges to some Funchsian group Γ_∞. We say that the pair $(\mathbb{H}/\Gamma_m, x_m)$ converges to $(\mathbb{H}/\Gamma_\infty, x_\infty)$ where x_∞ is the image of $i \in \mathbb{H}$ under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_\infty$. Let S_∞ be the hyperbolic surface of finite area whose connected components are the $\mathbb{H}/\Gamma_\infty$'s for different choices of base point x_m in different connected components of $S_m^{[c_0,\infty)}$. The surface S_∞ does not depend, up to isometry, on the choice of the base point x_m in a fixed connected component of $S_m^{[c_0,\infty)}$. One can check that $(S_m) \to S_\infty$ in $\overline{\mathcal{M}_{g,n}}$.

Convergence of functions

Fix an $\epsilon > 0$ and choose a base point $x_m \in S_m^{[\epsilon,\infty)}$ for each m. Assume that the pair $(\mathbb{H}/\Gamma_m, x_m)$ converges to $(\mathbb{H}/\Gamma_\infty, x_\infty)$ where, for each $m \in \mathbb{N} \cup \{\infty\}$, the point $i \in \mathbb{H}$ maps to x_m under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$.

For a C^∞ function f on S_m denote by $\widetilde{f}$ the lift of f under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$. Let (f_m) be a sequence of functions in $C^\infty(S_m) \cap L^2(S_m)$. One says that (f_m)*converges to* a continuous function f_∞ if $\widetilde{f_m}$ converges, uniformly over compact subsets of $\mathbb{H}$, to $\widetilde{f_\infty}$ for each choice of base points $x_m \in S_m^{[\epsilon,\infty)}$ and for each $\epsilon < \epsilon_0$ (ϵ_0 is the Margulis constant)

In the following, for a function $f \in L^2(S)$, we shall denote the L^2 norm of f by $\|f\|$. Also, for $f \in L^2(V)$ and $U \subset V$, we denote the L^2-norm of the restriction of f to U by $\|f\|_U$. A function f will be called *normalized* if $\|f\| = 1$. An eigenpair (λ, ϕ) will be called normalized if ϕ is normalized.

3.1.1 Results of G. Courtois-B. Colbois and D. Hejhal

We shall discuss the results due to G. Courtois-B. Colbois [C-C] and D. Hejhal [H]. These results consider behavior of eigenvalues that limits strictly below $\frac{1}{4}$.

G. Courtois-B. Colbois's result

Let (S_m) be a sequence in $\mathcal{M}_g$ that converges to S_∞ in $\overline{\mathcal{M}_g}$. Let S_∞ has k eigenvalues $0 = \lambda_0(S_\infty) = ... = \lambda_i(S_\infty) < \lambda_{i+1}(S_\infty) \leq ... \leq \lambda_{k-1}(S_\infty) < \frac{1}{4}$ where 0 appears as many times as the number of components of S_∞. For each $m \geq 1$ let S_m has k_m eigenvalues $0 = \lambda_0(S_m) < \lambda_1(S_m) \leq ... \leq \lambda_{k_m}(S_m) < \frac{1}{4}$. The result of Courtois-Colbois [C-C, THÉORÉM 0.1] is the following theorem.

Theorem 3.1.2. ***(Courtois-Colbois)***
For all m large, $k_m \geq k$. Moreover, for $1 \leq i \leq k$,

$$\lim_m \lambda_i(S_m) = \lambda_i(S_\infty) \quad \textit{and,} \quad \textit{for } \; k < i \leq k_m, \quad \lim_m \lambda_i(S_m) = \frac{1}{4}.$$

Authors carefully study the distribution of norms of eigenfunctions inside pinching tubes.

D. Hejhal's result

In [H] D. Hejhal considers a similar situation. Using the convergence of the Green's functions of these surfaces, he proves the following result.

Theorem 3.1.3. ***(Hejhal)*** *Let (S_m) be a sequence in $\mathcal{M}_g$ that converges to S_∞ in $\overline{\mathcal{M}_g}$. Let $\lambda_\infty < \frac{1}{4}$ be an eigenvalue of S_∞ with multiplicity k. Then there exists exactly k normalized eigenpairs $(\lambda_m^1, \phi_m^1), ..., (\lambda_m^k, \phi_m^k)$ of S_m such that $\lim_m \lambda_m^i = \lambda_\infty$, for each $1 \leq i \leq k$, and each ϕ_m^i converges uniformly over compacta to a normalized λ_∞-eigenfunction on S_∞.*

3.1.4 Results of L. Ji and S. Wolpert

Now we recall the results due to Lizhen Ji [Ji] and Scott Wolpert [Wo].

L. Ji's result

Lizhen Ji considers a family of closed hyperbolic surfaces (S_l) in $\mathcal{M}_g$ that converges to $S_\infty \in \overline{\mathcal{M}_g}$. His method of comparison of functions on different surfaces is a bit different than ours (the one discussed above). To compare the functions on the surfaces S_l and S_0, Ji considers the harmonic map of infinite energy $\pi_l : S_0 \to S_l$ constructed by M. Wolf [W]. The main theorem in [Ji] concerning convergence of eigenpairs is the following theorem that we quote using our set up i.e. we consider a sequence $(S_m) \in \mathcal{M}_g$ that converges to $S_\infty \in \overline{\mathcal{M}_g}$ and use the notion of convergence of functions defined in 3.1.

Theorem 3.1.5. ***([Ji, Theorem 1.2])*** *Let (λ_m, ϕ_m) be an eigenpair of S_m with L^2-norm equal to 1. Assume that $\lambda_m \to \lambda_\infty$ as $m \to \infty$.*
1. If $\|\phi_m\| \nrightarrow 0$ then, up to extracting a subsequence, (ϕ_m) converges to a non-zero λ_∞-eigenfunction ϕ_∞ on S_∞.
2. If $\|\phi_m\| \to 0$ then the followings hold:
(a) The limit $\lambda_\infty = \frac{1}{4} + t^2 \geq \frac{1}{4}$ for some $t \geq 0$.
(b) There exist constants $K_m \to \infty$ such that, up to extracting a subsequence, $(K_m \phi_{l_j})$ converges to a non-zero function ψ_∞ on S_∞ as $j \to \infty$.
(c) The function ψ_∞ satisfies $\Delta_0 \psi_\infty = (\frac{1}{4} + t^2)\psi_\infty$.
(d) There exists constants $a_1, ..., a_{2m}$ and possibly a λ_∞-cuspidal eigenfunction ϕ such that

$$\psi_\infty = \sum_{i=1}^{2m} a_i E_i(.; \frac{1}{2} + it) + \phi \tag{3.1}$$

where $E_i(.; \frac{1}{2} + it)$ $(1 \leq i \leq 2m)$ is the Eisenstein series associated to the i-th puncture of S_∞ and $2m$ is the total number of punctures of S_∞.
(e) If $\lambda_\infty = \frac{1}{4} + t^2$ is not an eigenvalue of S_∞, then

$$\psi_\infty = \sum_{i=1}^{2m} a_i E_i(.; \frac{1}{2} + it) \neq 0.$$

His proof involves estimates of distribution of mass $\|\phi_m\|$ of ϕ_m with respect to the thin/thin decomposition of S_m when $S_m \to S_\infty$.

S. Wolpert's result

The situation for Scott Wolpert is different. He focuses on sequence of generalized eigenpairs (λ_m, ϕ_m) of (S_m) when $\lambda_m \to \lambda_\infty < \infty$ and $\lambda_\infty > \frac{1}{4}$. Recall that a generalized eigenpair (λ_m, ϕ_m) is a pair such that

$$\Delta_m \phi_m = \lambda_m \phi_m$$

where ϕ_m satisfies *at most polynomial growth rate* in each cusp (see [Wo]). Main result of [Wo] concerning convergence behavior of ϕ_m is the following theorem that we quote with our notations.

Theorem 3.1.6. *([Wo, Theorem 3.4]) Let(λ_m, ϕ_m) be a generalized eigenpair of (S_m) such that $\lambda_m \to \lambda_\infty < \infty$ and $\lambda_\infty > \frac{1}{4}$. Let $S_m \to S_\infty$ in $\mathcal{M}_{g,n}$. There exists a constant c_0 such that if $\|\phi_m\|_{S_m{}^{[c_0,\infty)}}$ are bounded then for any α with $0 < \alpha < 1$, up to extracting a subsequence, (ϕ_m) converges to a generalized λ_∞-eigenfunction on S_∞ in the $C^{2,\alpha}$-topology. Furthermore for each c, $\lim \|\phi_m\|_{S_m{}^{[c,\infty)}} = \|\phi_\infty\|_{S_\infty{}^{[c,\infty)}}$.*

Scott Wolpert studies, like Lizhen Ji, the distribution of mass of eigenfunctions with respect to thick/thin decomposition of surfaces. Methods used, however, are quite different.

3.2 Convergence of small cuspidal eigenpairs

In this section we study the behavior of sequences of small cuspidal eigenpairs (λ_n, f_n) of $S_n \in \mathcal{M}_{g,n}$ when (S_n) converges to the surface $S_\infty \in \overline{\mathcal{M}_{g,n}}$. We prove the following theorem which has close resemblance with the results mentioned above. However, our result does not follow from any of those.

Theorem 3.2.1. *Let $S_m \to S_\infty$ in $\overline{\mathcal{M}_{g,n}}$. Let (λ_m, ϕ_m) be a normalized (L^2-norm of ϕ_m is 1) small cuspidal eigenpair of S_m. Assume that λ_m converges to λ_∞. Then one of the following holds:*
(1) *There exist strictly positive constants ϵ, δ such that $\limsup \|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$. Then (ϕ_m), up to extracting a subsequence, converges to a λ_∞-eigenfunction ϕ_∞ on S_∞.*
(2) *For each $\epsilon > 0$ the sequence $\|(\phi_m)\|_{S_m^{[\epsilon,\infty)}} \to 0$. Then, up to extracting a subsequence, (ϕ_m) converges to the zero function on S_∞. Moreover, there exists constants $K_{m_j} \to \infty$ such that, up to extracting a subsequence, $(K_m \phi_m)$ converges to a linear combination of Eisenstein series and (possibly) a cuspidal λ_∞-eigenfunction on S_∞.*
The later possibility arises only when: $S_\infty \in \partial\mathcal{M}_{g,n}$ and $\lambda_\infty = \frac{1}{4}$.

To prove Theorem 3.2.1 we need to understand how the *mass* (L^2 norm) of a small eigenfunction is distributed over the surface, and in particular how it is distributed with respect to the *thin/thick* decomposition which we explain now.

3.2.2 The thick / thin decomposition of a hyperbolic surface

Let $S \in \mathcal{M}_{g,n}$. Recall that for any $\epsilon > 0$, *the ϵ-thin part of S*, $S^{(0,\epsilon)}$, is the set of points of S with injectivity radius $< \epsilon$. The complement of $S^{(0,\epsilon)}$, *the ϵ-thick part of S*, denoted by $S^{[\epsilon,\infty)}$, is the set of points where the injectivity radius of S is $\geq \epsilon$. By Margulis lemma there exists a constant $\epsilon_0 > 0$, the Margulis constant, such that for all $\epsilon \leq \epsilon_0$, $S^{(0,\epsilon)}$ is a disjoint union of embedded collars, one for each geodesic of length less than 2ϵ, and of embedded cusps, one for each puncture. The collar around a geodesic of length $\leq \epsilon$ is called a *Margulis tube*. For definitions of collars and cusps we refer to Chapter 1.

3.3 Mass distribution of small cuspidal functions over the thin part

Our goal is to study the behavior of sequences of small cuspidal eigenpairs (λ_n, f_n) of $S_n \in \mathcal{M}_{g,n}$ when (S_n) converges to $S_\infty \in \overline{\mathcal{M}_{g,n}}$ and finally to prove Theorem 3.2.1. For this we need to understand how the *mass* (L^2 norm) of a small eigenfunction is distributed over the surface, and in particular how it is distributed with respect to the *thin/thick* decomposition. Let $S \in \mathcal{M}_{g,n}$. Recall that for any $\epsilon \leq \epsilon_0$ the ϵ-thin part, $S^{(0,\epsilon)}$, of S consists of cusps and Margulis tubes. We separately study the mass distribution of a small cuspidal eigenfunction over these two different types of domains.

3.3.1 Mass distribution over cusps

For $2\pi \leq a < b$ consider the annulus $\mathcal{P}(a,b) = \{(x,y) \in \mathcal{P}^1 : a \leq y < b\}$ contained in a cusp $\mathcal{P}^1$ and bounded by two horocycles of length $\frac{2\pi}{a}$ and $\frac{2\pi}{b}$. We begin our study with the following lemma.

Lemma 3.3.2. *For any $b > 2\pi$ there exists $K(b) < \infty$ such that for any small cuspidal eigenpair (λ, f) of $\mathcal{P}^1$ one has*

$$\|f\|_{\mathcal{P}(b,\infty)} < K(b)\|f\|_{\mathcal{P}(2\pi,b)}. \tag{3.2}$$

If $\lambda < \frac{1}{4} - \eta$ for some $\eta > 0$ then there exists a constant $T(b,\eta) < \infty$ depending on b and η such that for any small eigenpair (λ, f) one has

$$\|f\|_{\mathcal{P}(b,\infty)} < T(b,\eta)\|f\|_{\mathcal{P}(2\pi,b)}. \tag{3.3}$$

Furthermore, $K(b), T(b,\eta) \to 0$ as $b \to \infty$.

Proof. We begin with the first part. Since f is cuspidal inside $\mathcal{P}^1$ it can be expressed as

$$f(z) = \sum_{n \in \mathbb{Z}^*} f_n W_s(nz) \tag{3.4}$$

where $s(1-s) = \lambda$ and W_s is the *Whittaker function* (see [I, Proposition 1.5]). The meaning of (2) is that the right hand series converges to f in $L^2(\mathcal{P}^1)$ and that the convergence is uniform over compact subsets. Recall also that for $n \in \mathbb{Z}^*$ the Whittaker functions is defined by

$$W_s(nz) = 2(|n|y)^{\frac{1}{2}} K_{s-\frac{1}{2}}(|n|y)e^{inx}$$

where K_ϵ is the *McDonald's function* and that for any ϵ (see [Le, p. 119])

$$K_\epsilon(y) = \frac{1}{2}\int_{-\infty}^{+\infty} e^{-y\cosh u - \epsilon u} du \tag{3.5}$$

whenever the integral makes sense. From the expression it is clear that the functions $(W_s(n.))$ form an orthogonal family over $\mathcal{P}(a,b)$ (independent of the choices of a and b). Hence (1) will follow from the following claim.

Claim 3.3.3. *Let* $s \in [\frac{1}{2}, 1]$. *Then for any* $b > 2\pi$ *there exists* $K(b) < \infty$ *such that for all* $n \in \mathbb{Z}^*$

$$\|W_s(nz)\|_{\mathcal{P}(b,\infty)} \leq K(b)\|W_s(nz)\|_{\mathcal{P}(2\pi,b)}.$$

Furthermore, $K(b) \to 0$ *as* $b \to \infty$.

Proof. From the expression of W_s we have

$$\|W_s(nz)\|_{\mathcal{P}(a,b)} = 2\pi\Big(\int_a^b 4|n|yK_{s-\frac{1}{2}}(|n|y)^2 \frac{dy}{y^2}\Big).$$

To prove the claim we may suppose that $n \geq 1$. Our next objective is to obtain bounds for the functions $K_{s-\frac{1}{2}}(y)$ for $s \in [\frac{1}{2}, 1]$. We start from the above integral representation of $K_\epsilon(y)$. We write $K_\epsilon(y) = \dfrac{1}{2}\{c(\epsilon, y) + d(\epsilon, y)\}$ where

$$c(\epsilon, y) = \int_{-1}^{1} e^{-y\cosh u - \epsilon u} du \tag{3.6}$$

and

$$d(\epsilon, y) = \int_{-\infty}^{-1} e^{-y\cosh u - \epsilon u} du + \int_{1}^{\infty} e^{-y\cosh u - \epsilon u} du. \tag{3.7}$$

Now we treat $c(\epsilon, y)$ and $d(\epsilon, y)$ separately.

<u>Bounding $c(\epsilon, y)$</u>:

We have

$$c(\epsilon, y) = \int_{-1}^{1} e^{-y\cosh u}.e^{-\epsilon u} du \leq e^{\epsilon}.\int_{-1}^{1} e^{-y\cosh u} du = e^{\epsilon}\int_{-1}^{1} e^{-y(1+\frac{u^2}{2!}+\frac{u^4}{4!}+\ldots)} du$$

$$= e^{\epsilon}.e^{-y}\int_{-1}^{1} e^{-y(\frac{u^2}{2!}+\frac{u^4}{4!}+\ldots)} du \leq 2e^{\epsilon}.e^{-y}\int_{0}^{1} e^{-y\frac{u^2}{2!}} du.$$

Since $e^{\frac{yu^2}{2}} > 1 + \frac{yu^2}{2}$ for $u > 0$, we have:

$$\int_0^1 e^{-y\frac{u^2}{2!}} du < \int_0^1 \frac{du}{1 + \frac{yu^2}{2}} = \frac{2}{y}\tan^{-1}(\frac{y}{2}) \leq \frac{2}{y}.\frac{\pi}{2}.$$

Therefore

$$c(\epsilon, y) \leq 2\pi e^{\epsilon}\frac{e^{-y}}{y}.$$

To obtain a lower bound, we write

$$\int_{-1}^{1} e^{-y\cosh u}.e^{-\epsilon u} du \geq e^{-\epsilon}.\int_{-1}^{1} e^{-y\cosh u} du = e^{-\epsilon}\int_{-1}^{1} e^{-y(1+\frac{u^2}{2!}+\frac{u^4}{4!}+\ldots)} du$$

$$= 2e^{-\epsilon}.e^{-y}\int_{0}^{1} e^{-y(\frac{u^2}{2!}+\frac{u^4}{4!}+\ldots)} du.$$

Since for all $u \in (0, 1]$ one has

$$\frac{u^2}{2!} + \frac{u^4}{4!} + \ldots < u(\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \ldots) = u.$$

Hence

$$c(\epsilon,y) \geq 2e^{-\epsilon}.e^{-y}\int_0^1 e^{-uy}du = 2e^{-\epsilon}\frac{e^{-y}}{y}(1-e^{-y}).$$

Combining the above two inequalities

$$2e^{-\epsilon}\frac{e^{-y}}{y}(1-e^{-y}) \leq c(\epsilon,y) \leq 2\pi e^{\epsilon}\frac{e^{-y}}{y}.$$

<u>Bounding $d(\epsilon,y)$</u>:

$$\begin{aligned} d(\epsilon,y) &= \int_{-\infty}^{-1} e^{-y\cosh u-\epsilon u}du + \int_1^\infty e^{-y\cosh u-\epsilon u}du \\ &= \int_1^\infty e^{-y\cosh u-\epsilon u}du + \int_1^\infty e^{-y\cosh u+\epsilon u}du. \end{aligned}$$

Now for any $u>1$,

$$\frac{u^2}{2!}+\frac{u^4}{4!}+... > \gamma_0 u^2 > \gamma_0 u$$

where $\gamma_0 = \sum_{n=1}^{\infty}\frac{1}{(2n)!}$.

Thus

$$\begin{aligned} d(\epsilon,y) &= e^{-y}\int_1^\infty \{e^{-y(\frac{u^2}{2!}+\frac{u^4}{4!}+...)-\epsilon u} + e^{-y(\frac{u^2}{2!}+\frac{u^4}{4!}+...)+\epsilon u}\}du \\ &\leq e^{-y}\int_1^\infty \{e^{-y\gamma_0 u-\epsilon u} + e^{-y\gamma_0 u+\epsilon u}\}du \\ &= \frac{e^{-y}}{y}\left(\frac{e^{-(y\gamma_0+\epsilon)}}{\gamma_0+\frac{\epsilon}{y}} + \frac{e^{-(y\gamma_0-\epsilon)}}{\gamma_0-\frac{\epsilon}{y}}\right). \end{aligned}$$

Thus combining the estimates for $c(\epsilon,y)$ and $d(\epsilon,y)$ we obtain

$$2e^{-\epsilon}\frac{e^{-y}}{y}(1-e^{-y}) < K_\epsilon(y) < 2\pi e^{\epsilon}\frac{e^{-y}}{y} + \frac{e^{-y}}{y}\left(\frac{e^{-(y\gamma_0+\epsilon)}}{\gamma_0+\frac{\epsilon}{y}} + \frac{e^{-(y\gamma_0-\epsilon)}}{\gamma_0-\frac{\epsilon}{y}}\right).$$

Let

$$\delta(\epsilon,y) = \frac{e^{-(y\gamma_0+\epsilon)}}{\gamma_0+\frac{\epsilon}{y}} + \frac{e^{-(y\gamma_0-\epsilon)}}{\gamma_0-\frac{\epsilon}{y}}.$$

Observe that for $\epsilon<1$ and $y \geq \frac{2}{\gamma_0}$

$$\delta(\epsilon,y) < \frac{4\cosh 1}{\gamma_0}e^{-\gamma_0 y} = \delta_0(y).$$

So, for $y \geq \frac{2}{\gamma_0}$ large enough

$$2e^{-\epsilon}\frac{e^{-y}}{y} < K_\epsilon(y) < \frac{e^{-y}}{y}\left(2\pi e^{\epsilon}+\delta_0(y)\right). \tag{3.8}$$

Going back to the expression of W_s, for $s\in[\frac{1}{2},1]$, we find:

$$\frac{1}{2\pi}\|W_s(nz)\|^2_{\mathcal{P}(2\pi,b)} = \int_{2\pi}^b 4nyK_{s-\frac{1}{2}}(ny)^2\frac{dy}{y^2} = \int_{2\pi}^b 4nK_{s-\frac{1}{2}}(ny)^2\frac{dy}{y}$$

$$\geq \int_{2\pi}^{b} \frac{4n}{b} K_{s-\frac{1}{2}}(ny)^2 dy > \frac{16ne^{1-2s}}{b} \int_{2\pi}^{b} \frac{e^{-2ny}}{(ny)^2} dy = \frac{16ne^{1-2s}}{n^2 b} \int_{2\pi}^{b} \frac{e^{-2ny}}{y^2} dy$$

$$= \frac{16ne^{1-2s}}{n^2 b} \left(\int_{2\pi}^{\frac{b}{2}} \frac{e^{-2ny}}{y^2} dy + \int_{\frac{b}{2}}^{b} \frac{e^{-2ny}}{y^2} dy \right) > \frac{16ne^{1-2s}}{n^2 b} \left(\int_{\frac{b}{2}}^{b} \frac{e^{-2ny}}{y^2} dy \right)$$

$$= \frac{16e^{1-2s}}{nb} \frac{e^{-nb}}{n\frac{b^2}{4}} \{1 + O(e^{-nb} + \frac{2}{b})\}$$

i.e.

$$\|W_s(nz)\|^2_{\mathcal{P}(2\pi,b)} > 2\pi \frac{16e^{1-2s}}{nb} \frac{e^{-nb}}{n\frac{b^2}{4}} \{1 + O(e^{-nb} + \frac{1}{b})\} \tag{3.9}$$

Also,

$$\frac{1}{2\pi} \|W_s(nz)\|^2_{\mathcal{P}(b,\infty)} = \int_b^\infty 4ny K_{s-\frac{1}{2}}(ny)^2 \frac{dy}{y^2} = \int_b^\infty 4n K_{s-\frac{1}{2}}(ny)^2 \frac{dy}{y}$$

$$\leq \int_b^\infty \frac{4n}{b} K_{s-\frac{1}{2}}(ny)^2 dy \leq \frac{4n(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2}{b} \int_b^\infty \frac{e^{-2ny}}{(ny)^2} dy$$

$$= \frac{4(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2}{nb} \frac{e^{-2nb}}{2nb^2} \{1 + O(\frac{1}{b})\}$$

i.e.

$$\|W_s(nz)\|^2_{\mathcal{P}(b,\infty)} \leq 2\pi \frac{2(2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2}{nb} \frac{e^{-2nb}}{nb^2} \{1 + O(\frac{1}{b})\} \tag{3.10}$$

In the last inequality, we used the following estimate from [Le, Section 3.2]:

$$\int_{t_1}^{t_2} \frac{e^{-2\alpha y}}{y^2} dy = \frac{e^{-2\alpha t_1}}{2\alpha {t_1}^2} \{1 + O(e^{2(t_1 - t_2)} + {t_1}^{-1})\}$$

with an absolute constant for the O-term for $\alpha > 1$.

Comparing (3.9) and (3.10) we get, for any $n \in \mathbb{Z}^*$

$$\|W_s(nz)\|_{\mathcal{P}(b,\infty)} \leq K(b) \|W_s(nz)\|_{\mathcal{P}(2\pi,b)} \tag{3.11}$$

where

$$K^2(b) = \frac{e^{2s-1}}{8} (2\pi e^{(s-\frac{1}{2})} + \delta_0(b))^2 e^{-|n|b} \frac{(1 + O(\frac{1}{b}))}{1 + O(e^{-|n|b} + \frac{2}{b})}.$$

From the expression it is clear that K is bounded independent of n, b (once b is large enough) and $s \in [\frac{1}{2}, 1]$. So we obtain the claim by choosing some $b > \frac{2}{\gamma_0}$ sufficiently large (once and for all) such that the O-terms in the expression of T are small enough. It is also clear from the expression that when $b \to \infty$, $K(b) \to 0$. This proves the Claim 3.3.3 and hence the first part of Lemma 3.3.2.

Now we prove the second part. Let $\lambda < \frac{1}{4} - \eta$ for some $\eta > 0$ and let (λ, f) be a residual eigenpair. The Fourier expansion of f inside $\mathcal{P}^1$ has the form

$$f(z) = f_0 y^s + \sum_{n \in \mathbb{Z}^*} f_n W_s(nz) = f_0 y^s + g(z) \tag{3.12}$$

where $s(1-s)=\lambda$, $s\in(0,\frac{1}{2})$ (see [I]) and $g(z)=\sum_{n\in\mathbb{Z}^*} f_n W_s(nz)$. Since $f_0 y^s$ and g are orthogonal and since the first part can be applied to g, one needs only to prove the lemma for the term $f_0 y^s$. So we calculate:

$$\int_a^c y^{2s}\frac{dy}{y^2}=\frac{1}{1-2s}\left(\frac{1}{a^{1-2s}}-\frac{1}{c^{1-2s}}\right).$$

Therefore, for $b>2\pi$,

$$\|f_0 y^s\|^2_{\mathcal{P}(b,\infty)}=\frac{1}{(\frac{b}{2\pi})^{1-2s}-1}\|f_0 y^s\|^2_{\mathcal{P}(2\pi,b)}. \tag{3.13}$$

The lemma is satisfied by $T_2(b,\eta)$ such that

$$T_2^2(b,\eta)=\max\left(K^2(b),\frac{1}{(\frac{b}{2\pi})^{1-2s}-1}\right).$$

From the expression it is clear that $T_2(b,\eta)$ depends only on two quantities: b and $\frac{1}{2}-s$. Since $\frac{1}{2}-s>\sqrt{\eta}>0$, $\frac{1}{(\frac{b}{2\pi})^{1-2s}-1}\to 0$ when $b\to\infty$. This proves the second part.

3.3.4 Mass distribution over Margulis tubes

Now we study the distribution of the mass of a small eigenfunction over Margulis tubes. Let γ be a simple closed geodesic of length $l_\gamma=2\pi l$. Recall that $\mathcal{C}^a$ denotes the collar around γ bounded by two equidistant curves of length a. Any $f\in L^2(\mathcal{C}^1)$ can be written as a Fourier series in the θ-coordinate:

$$f(r,\theta)=a_0(r)+\sum_{j=1}^{\infty}\left(a_j(r)\cos j\theta+b_j(r)\sin j\theta\right). \tag{3.14}$$

The functions $a_j=a_j(r)$ and $b_j=b_j(r)$ are defined on $[-\cosh^{-1}(\frac{1}{l_\gamma}),\cosh^{-1}(\frac{1}{l_\gamma})]$ and are called the *j-th Fourier coefficients of f* (in $\mathcal{C}^1$). When f is a λ-eigenfunction, a_j and b_j are solutions of the differential equation

$$\frac{d^2\phi}{dr^2}+\tanh r\frac{d\phi}{dr}+(\lambda-\frac{j^2}{l^2\cosh^2 r})\phi=0. \tag{3.15}$$

We set $[f]_0=a_0(r)$ and $[f]_1=f-[f]_0$. The following lemma concerns the distribution of masses of $[f]_0$ and $[f]_1$ inside $\mathcal{C}^1$.

Lemma 3.3.5. *For any $l_\gamma<\epsilon\le\epsilon_0$ there exist constants $T_1(\epsilon),T_2(\epsilon)<\infty$, depending only on ϵ, such that for any small eigenpair (λ,f) of $\mathcal{C}^1$ the following inequalities hold:*

$$\|[f]_1\|_{\mathcal{C}^\epsilon}<T_1(\epsilon)\|[f]_1\|_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon} \tag{3.16}$$

and

$$\|[f]_0\|_{\mathcal{C}^{\epsilon_0}\setminus\mathcal{C}^\epsilon}<T_2(\epsilon)\|[f]_0\|_{\mathcal{C}^1\setminus\mathcal{C}^{\epsilon_0}}. \tag{3.17}$$

Therefore, for any $l_\gamma<\epsilon\le\epsilon_0$ and any small eigenpair (λ,f) of $\mathcal{C}^1$ one has

$$\|f\|_{\mathcal{C}^{\epsilon_0}\setminus\mathcal{C}^\epsilon}<\mathit{max}\,\{T_1(\epsilon_0),T_2(\epsilon)\}\|f\|_{\mathcal{C}^1\setminus\mathcal{C}^{\epsilon_0}}. \tag{3.18}$$

If $\lambda<\frac{1}{4}-\eta$ for some $\eta>0$ then there exists a constant $T_0(\epsilon,\eta)<\infty$, depending only on η and ϵ, such that

$$\|[f]_0\|_{\mathcal{C}^\epsilon}<T_0(\epsilon,\eta)\|[f]_0\|_{\mathcal{C}^1\setminus\mathcal{C}^\epsilon}. \tag{3.19}$$

Furthermore, $T_1(\epsilon),T_0(\epsilon,\eta)\to 0$ as $\epsilon\to 0$.

Before starting the proof of the above lemma we make a few observations about the solutions of (3.15). The change of variable $u(r) = \cosh^{\frac{1}{2}}(r)\phi(r)$ transforms (3.15) into

$$\frac{d^2u}{dr^2} = \Big((\frac{1}{4} - \lambda) + \frac{1}{4\cosh^2 r} + \frac{j^2}{l^2\cosh^2 r} \Big) u. \tag{3.20}$$

Let s_j (resp. c_j) be the solution of (3.20) satisfying the conditions: $s_j(0) = 0$ and $s_j^{'}(0) = 1$ (resp. $c_j(0) = 1$ and $c_j^{'}(0) = 0$). Since (3.20) is invariant under $r \to -r$ one has: $s_j(-r) = -s_j(r)$ and $c_j(-r) = c_j(r)$ for all $j \geq 0$. Therefore there exists $t > 0$ such that $s_j > 0$ and $c_j^{'} > 0$ on $(0, t]$. Now we prove the following claim.

Claim 3.3.6. *Let $L > 0$. Let $g : [0, L] \to \mathbb{R}$ be a C^2-function which satisfies the inequality:*

$$\frac{d^2g}{dr^2} > \delta^2 g$$

for some $\delta > 0$. If $g^{'}(0) \geq 0$ then $\dfrac{g(r)}{\cosh \delta r}$ is a monotone increasing function of r in $(0, L]$.

Proof. Observe that

$$\Big(\frac{g(r)}{\cosh \delta r}\Big)^{'} = \frac{g^{'}(r)\cosh \delta r - \delta g(r) \sinh \delta r}{\cosh^2(\delta r)}.$$

Consider the function H defined on $[0, L]$ by

$$H(r) = g^{'}(r)\cosh \delta r - \delta g(r) \sinh \delta r.$$

Since g is a C^2 function H is continuous on $[0, L]$. Observe that the claim follows if $H(r) > 0$ in $(0, L]$. Now for any $r \in (0, L]$

$$H^{'}(r) = g^{''}(r)\cosh \delta r - \delta^2 g(r)\cosh \delta r = (g^{''}(r) - \delta^2 g(r))\cosh \delta r > 0.$$

Therefore for $r > 0$, $H(r) > H(0) = g^{'}(0) \geq 0$. Hence the claim.□

Proof of Lemma 3.3.5. We need to estimate, for $l_\gamma \leq t < w \leq 1$, the quantities:

$$\|[f]_1\|^2_{\mathcal{C}^w \setminus \mathcal{C}^t} = l_\gamma \int_{-L_w}^{-L_t} \Big(\sum_{j=1}^{\infty} \alpha_j^2 + \beta_j^2 \Big) dr + l_\gamma \int_{L_t}^{L_w} \Big(\sum_{j=1}^{\infty} \alpha_j^2 + \beta_j^2 \Big) dr$$

and

$$\|[f]_0\|^2_{\mathcal{C}^w \setminus \mathcal{C}^t} = l_\gamma \int_{-L_w}^{-L_t} \alpha_0^2 dr + l_\gamma \int_{L_t}^{L_w} \alpha_0^2 dr$$

where $\alpha_0(r) = \cosh^{\frac{1}{2}}(r) a_0(r)$, $\alpha_j(r) = a_j(r)\cosh^{\frac{1}{2}}(r)$, $\beta_j(r) = b_j(r)\cosh^{\frac{1}{2}}(r)$ and $L_u = \cosh^{-1}(\frac{u}{l_\gamma})$. Since s_j is odd and c_j is even, for any symmetric subset $U \subset [-L_1, L_1]$, s_j and c_j are orthogonal in $L^2(U)$. Now α_j and β_j are linear combinations of s_j and c_j for $j \geq 1$ and α_0 is a linear combination of s_0 and c_0. Therefore, since s_j and c_j are orthogonal, it is enough to prove the lemma with s_j and c_j instead of $[f]_1$ and with s_0 and c_0 instead of $[f]_0$. We detail the computations for s_j. The computations for c_j are similar. Let us choose ϵ such that $l_\gamma < \epsilon < \epsilon_0$. The lemma reduces to find $K_1(\epsilon), K_2(\epsilon) < \infty$, depending on ϵ, and $K_0(\epsilon, \eta) < \infty$, depending on ϵ, η (> 0), such that

$$\|s_j\|_{\mathcal{C}^\epsilon} < K_1(\epsilon)\|s_j\|_{\mathcal{C}^1 \setminus \mathcal{C}^\epsilon}, \qquad \|s_0\|_{\mathcal{C}^{\epsilon_0} \setminus \mathcal{C}^\epsilon} < K_2(\epsilon)\|s_0\|_{\mathcal{C}^1 \setminus \mathcal{C}^{\epsilon_0}}$$

and

$$\|s_0\|_{\mathcal{C}^\epsilon} < K_0(\epsilon, \eta)\|s_0\|_{\mathcal{C}^1 \setminus \mathcal{C}^\epsilon}.$$

Let $\eta < \frac{1}{4} - \lambda$ and set $\delta_0 = \sqrt{\eta}$ and set for $j \geq 1$, $\delta_j = 1$. Notice that $l \cosh r < 1$ on $[0, L_1)$. Hence by (3.20) $s_j : [0, L_1) \to \mathbb{R}$ satisfies the inequality:

$$\frac{d^2 s_j}{dr^2} > \delta_j^2 s_j.$$

Hence by Claim 3.3.6 $h_j(r) = \frac{s_j(r)}{\cosh r}$, for $j \geq 1$, is strictly increasing on $(0, L_1)$. The same is true for $h_0 = \frac{s_0(r)}{\cosh \delta_0 r}$ (even when $\delta_0 = 0$).

We begin with the proof of the second part of the Lemma. So we assume $\eta > 0$. For $0 \leq a < b$ consider the integral:

$$\int_a^b s_0^2(r)dr = \int_a^b h_0^2(r)\cosh^2(\delta_0 r)dr.$$

Since h_0 is strictly increasing we have

$$h_0^2(a)\int_a^b \cosh^2(\delta_0 r)dr < \int_a^b s_0^2(r)dr < h_0^2(b)\int_a^b \cosh^2(\delta_0 r)dr. \tag{3.21}$$

Now choosing $a = 0$ and $b = L_\epsilon$ the last inequality in (3.21) gives

$$\|s_0\|^2_{\mathcal{C}^\epsilon} < 2l_\gamma h_0^2(L_\epsilon)\int_0^{L_\epsilon} \cosh^2(\delta_0 r)dr. \tag{3.22}$$

Next choosing $a = L_\epsilon$ and $b = L_1$ the first inequality in (3.21) gives

$$\|s_0\|^2_{\mathcal{C}^1 \setminus \mathcal{C}^\epsilon} > 2l_\gamma h_0^2(L_\epsilon)\int_{L_\epsilon}^{L_1} \cosh^2(\delta_0 r)dr. \tag{3.23}$$

Therefore

$$\|s_0\|_{\mathcal{C}^\epsilon} < T_0\|s_0\|_{\mathcal{C}^1 \setminus \mathcal{C}^\epsilon} \tag{3.24}$$

where

$$T_0^2 = \frac{\sinh 2\delta_0 L_\epsilon + 2\delta_0 L_\epsilon}{\sinh 2\delta_0 L_1 - \sinh 2\delta_0 L_\epsilon + 2\delta_0(L_1 - L_\epsilon)}. \tag{3.25}$$

We see that T_0 depends only on ϵ, δ_0 and l_γ. Now $L_\epsilon = \cosh^{-1}(\frac{\epsilon}{l_\gamma}) = \log(\frac{\epsilon}{l_\gamma} + \sqrt{(\frac{\epsilon}{l_\gamma})^2 - 1})$. Therefore, for ϵ and $\delta_0^2 = \eta > 0$ fixed, and l_γ small

$$T_0^2 < K_0 \frac{1}{\epsilon^{-2\delta_0} - 1},$$

and the constant K_0 is independent of l_γ as soon as l_γ is small compared to ϵ. Thus we can choose $T_0(\epsilon, \eta)$ independent of l_γ satisfying (3.24). This proves (3.19)

For s_j, $j \geq 1$, exactly the same computations for s_0 work with δ_0 replaced by $\delta_j = 1$. Hence in this case our constant,

$$T_1^2(\epsilon) < K_1 \frac{1}{\epsilon^{-2} - 1},$$

depends only on ϵ. This proves (3.16).

Now we prove (3.18). Since $s_0 : [0, L_1] \to \mathbb{R}^+$ is strictly increasing we have:

$$\int_{L_\epsilon}^{L_{\epsilon_0}} s_0^2(r)dr < s_0^2(L_{\epsilon_0})(L_{\epsilon_0} - L_\epsilon) \text{ and } \int_{L_{\epsilon_0}}^{L_1} s_0^2(r)dr > s_0^2(L_{\epsilon_0})(L_1 - L_{\epsilon_0}).$$

Combining the two inequalities we obtain

$$\|s_0\|_{\mathcal{C}^{\epsilon_0}\setminus\mathcal{C}^{\epsilon}} < T_2(\epsilon)\|s_0\|_{\mathcal{C}^1\setminus\mathcal{C}^{\epsilon_0}} \tag{3.26}$$

where

$$T_2^2(\epsilon) = \frac{L_{\epsilon_0} - L_\epsilon}{L_1 - L_{\epsilon_0}} < K_2\left(\frac{\log\frac{1}{\epsilon}}{\log\frac{1}{\epsilon_0}} - 1\right). \tag{3.27}$$

The constant K_2 is independent of l_γ as soon as l_γ is small compared to ϵ. Thus we can choose $T_2(\epsilon)$ independent of l_γ satisfying (3.26). This proves (3.18).

3.3.7 Applications

Let S be a finite area hyperbolic surface with n punctures. Denote by $\mathcal{P}_i$ the standard cusp around the i-th puncture. Recall that $\mathcal{P}_i$'s have disjoint interiors and that each of them is isometric to the half-infinite annulus $\mathcal{P}^1$ (see 1.1.2). Applying Lemma 3.3.2 in each $\mathcal{P}_i$ separately we obtain the following corollary which will be useful in our analysis.

Corollary 3.3.8. *For any $0 < \epsilon < \epsilon_0$ there exists $T(\epsilon) < \infty$, depending only on ϵ, such that for any small cuspidal eigenpair (λ, f) of S one has*

$$\|f\|_{S_c^{(0,\epsilon)}} < T(\epsilon)\|f\|_{S_c^{(0,1]}\setminus S_c^{(0,\epsilon]}}. \tag{3.28}$$

If $\lambda < \frac{1}{4} - \eta$ for some $\eta > 0$ then for any $0 < \epsilon < \epsilon_0$ there exists $T_1(\epsilon, \eta) < \infty$, depending only on ϵ and η, such that for any λ-eigenfunction f of S one has

$$\|f\|_{S_c^{(0,\epsilon)}} < T_1(\epsilon, \eta)\|f\|_{S_c^{(0,1]}\setminus S_c^{(0,\epsilon]}}. \tag{3.29}$$

Furthermore, $T(\epsilon)$ and $T_1(\epsilon, \eta)$ tends to zero as $\epsilon \to 0$.

Using this corollary and (3.18) we deduce the following

Corollary 3.3.9. *For any $0 < \epsilon < \epsilon_0$ there exists a constant $L(\epsilon) < \infty$, depending only on ϵ, such that for any small cuspidal eigenfuction f of S one has*

$$\|f\|_{S^{[\epsilon,\infty)}} < L(\epsilon)\|f\|_{S^{[\epsilon_0,\infty)}}. \tag{3.30}$$

Now we give a new proof of the following theorem of D. Hejhal [H].

Theorem 3.3.10. *Consider a sequence $(S_m) \in \mathcal{M}_{g,n}$ converging to $S_\infty \in \overline{\mathcal{M}_{g,n}}$. Let (λ_m, ϕ_m) be a normalized small eigenpair of S_m such that $\lambda_m \to \lambda_\infty$. If $\lambda_\infty < \frac{1}{4}$ then, up to extracting a subsequence, ϕ_m converges to a normalized λ_∞-eigenfunction ϕ_∞ of S_∞.*

D. Hejhal's proof uses *convergence of Green's functions* of S_m to that of S_∞. Our approach is more elementary and uses the above estimates on the mass distribution of eigenfunctions over thin part of surfaces.

Proof of Theorem 3.3.10. First we prove that, up to extracting a subsequence, ϕ_m converges to a λ_∞-eigenfunction ϕ_∞ of S_∞. By Theorem 3.2.1 (which will be proven in §3) it is enough to prove that there exist $\epsilon, \delta > 0$ such that $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$ up to extracting a subsequence. We argue by contradiction. Suppose that for any $\epsilon > 0$ the sequence $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \to 0$ as $m \to \infty$. Let $\eta > 0$, such that $\lambda_m < \frac{1}{4} - \eta$ for all $m \geq 1$. By Lemma 3.3.5 we have

$$\|\phi_m\|_{\mathcal{C}^\epsilon} < \max\{T_0(\epsilon, \eta), T_1(\epsilon)\}\|\phi_m\|_{\mathcal{C}^1 \setminus \mathcal{C}^\epsilon}. \tag{3.31}$$

Therefore from (3.29) and (3.31) we have

$$\|\phi_m\|_{S_m^{(0,\epsilon)}} < \max\{T_0(\epsilon, \eta), T_1(\epsilon), T_1(\epsilon, \eta)\}\|\phi_m\|_{S_m^{[\epsilon,\infty)}}. \tag{3.32}$$

Hence if $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \to 0$ as $m \to \infty$ then $\|\phi_m\| \to 0$ as $m \to \infty$. This is a contradiction to the fact that each ϕ_m is normalized i.e. $\|\phi_m\| = 1$.

Next we prove that $\|\phi_\infty\| = 1$. By uniform convergence over compacta, in each cusp and in each pinching collar, the Fourier coefficients of ϕ_m will converge to the corresponding Fourier coefficients of ϕ_∞. Therefore, by (3.16), (3.19) and (3.29), ϕ_m's are uniformly integrable: for any $\delta > 0$ there exist $\epsilon > 0$ such that for all large values of m

$$\|\phi_m\|_{S_m^{[\epsilon,\infty)}} > 1 - \delta. \tag{3.33}$$

Hence $\|\phi_\infty\| = 1$. This finishes the proof. □

3.4 Proof of Theorem 3.2.1

Let (S_m) be a sequence in $\mathcal{M}_{g,n}$ which converges in $\overline{\mathcal{M}_{g,n}}$ to S_∞. Let Γ_m, Γ_∞ be such that $S_m = \mathbb{H}/\Gamma_m$ and $S_\infty = \mathbb{H}/\Gamma_\infty$. Recall that the convergence $S_m \to S_\infty$ means that for any fixed positive constant $\epsilon_1 \leq \epsilon_0$ (ϵ_0 is the Margulis constant) and a choice of base point $p_m \in S_m^{[\epsilon_1,\infty)}$, after conjugating Γ_m so that the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$ maps i to p_m, $(\mathbb{H}/\Gamma_m, p_m)$ converges to a component $(\mathbb{H}/\Gamma_\infty, p_\infty)$ of S_∞. We begin by fixing some $\epsilon < \epsilon_0$ and $p_m \in S_m^{[\epsilon,\infty)}$. In the following we assume that ϵ_1, p_m, Γ_m, p_∞ and Γ_∞ satisfy the previous statement.

To simplify notations we shall assume that only one closed geodesic γ_m gets pinched as $S_m \to S_\infty \in \partial\mathcal{M}_{g,n}$. In particular the limit surface S_∞ (which may be disconnected) has two new cusps. Denote the standard cusps of S_m by $\mathcal{P}_1(m), \mathcal{P}_2(m), ..., \mathcal{P}_n(m)$ and the limits of these in $S_\infty \in \partial\mathcal{M}_{g,n}$ by $\mathcal{P}_1(\infty)$,...,$\mathcal{P}_n(\infty)$ and denote by $\mathcal{P}_{n+1}(\infty), \mathcal{P}_{n+2}(\infty)$ the *new cusps* which arise due to the pinching of γ. The cusps $\mathcal{P}_i(\infty)$ for $1 \leq i \leq n$ will be called *old cusps*.

Recall that we have a sequence of small cuspidal eigenpairs (λ_m, ϕ_m) of $S_m = \mathbb{H}/\Gamma_m$ such that the L^2-norm of ϕ_m is 1 and $\lambda_m \to \lambda_\infty \leq \frac{1}{4}$.

Notation 3.4.1. *In what follows $d\mu_m$ will denote the area measure on S_m for $m \in \mathbb{N} \cup \{\infty\}$ and $d\mu_{\mathbb{H}}$ will denote the area measure on $\mathbb{H}$. The lift of $f \in L^2(S_m)$ to $\mathbb{H}$ under the projection $\mathbb{H} \to \mathbb{H}/\Gamma_m$, defined as above, will be denoted by $\widetilde{f}$.*

By Green's formula one has:

$$\int_{S_m} |\nabla\phi_m|^2 d\mu_m = \lambda_m \int_{S_m} |\phi_m|^2 d\mu_m = \lambda_m.$$

Let $K \subset \mathbb{H}$ be compact. One can cover K by finitely many geodesic balls of radius ρ. If ρ is sufficiently small then each of these balls maps injectively to S_m since $\Gamma_m \to \Gamma_\infty$. Therefore, since $\|\phi_m\| = 1$ $\|\widetilde{\phi_m}|_K\|$ is bounded depending only on K. From the mean value formula of Fay (see Chapter 1) there exists a constant $\Lambda(\lambda_\infty, \rho)$ such that for λ_m close to λ_∞,

$$|\widetilde{\phi_m}(q)| \leq \Lambda(\lambda_\infty, \rho) \int_{N(K, \frac{\rho}{2})} |\widetilde{\phi_m}| d\mu_{\mathbb{H}}$$

for each $q \in K$ where $N(K, r)$ denotes the closed neighborhood of radius r of K in $\mathbb{H}$. Next we use the gradient bound for eigenfunctions of the Laplacian (see Chapter 1) to obtain a uniform bound for $\nabla\widetilde{\phi_m}$ on $N(K, \frac{\rho}{2})$. This makes $(\widetilde{\phi_m}|_K)$ an equicontinuous family. So, by Arzela-Ascoli theorem, up to extracting a subsequence, $(\widetilde{\phi_m})$ converges to a continuous function $\widetilde{\phi_\infty}$ on K. By a diagonalization argument one may suppose that the sequence works for all compact subsets of $\mathbb{H}$. Therefore, up to extracting a subsequence, $\widetilde{\phi_m} \to \widetilde{\phi_\infty}$ uniformly over compacta. By this uniform convergence it is clear that $\widetilde{\phi_\infty}$ is a *weak solution* of the Laplace equation: $\Delta u = \lambda_\infty u$. Therefore, by elliptic regularity, $\widetilde{\phi_\infty}$ is indeed smooth and satisfies

$$\Delta\widetilde{\phi_\infty} = \lambda_\infty \widetilde{\phi_\infty}.$$

Also by the convergence $\widetilde{\phi_\infty}$ induces a function ϕ_∞ on S_∞ that satisfies

$$\Delta\phi_\infty = \lambda_\infty \phi_\infty.$$

However, ϕ_∞ may not be an eigenfunction since it could be the zero function. In order to discuss this point, we shall consider two cases according to whether the L^2-norm $\|\phi_m\|_{S_m^{[\epsilon,\infty)}}$ of the restriction of ϕ_m to $S_m^{[\epsilon,\infty)}$ is bounded below by a positive constant or not.

Case 1: $\exists\ \epsilon, \delta > 0$ **such that** $\limsup \|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$**.**

We may assume that $\lim \|\phi_m\|_{S_m^{[\epsilon,\infty)}} \geq \delta$. Then by the uniform convergence of $\widetilde{\phi_m} \to \widetilde{\phi_\infty}$ over compacta,

$$\int_{S_\infty^{[\epsilon,\infty)}} \phi_\infty^2 d\mu_\infty = \lim_{m_j \to \infty} \int_{S_{m_j}^{[\epsilon,\infty)}} \phi_{m_j}^2 d\mu_{m_j} \geq \delta > 0.$$

Therefore ϕ_∞ is not the zero function and its L^2 norm is less than 1. Therefore it is a λ_∞-eigenfunction.

Case 2: For any $\epsilon > 0$ **the sequence** $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \to 0$**.**

Then we will prove the following statements:
(i) $S_\infty \in \partial\mathcal{M}_{g,n}$,
(ii) $\lambda_\infty = \frac{1}{4}$ and
(iii) $\exists$ constants K_m such that, up to extracting a subsequence, $(K_m\widetilde{\phi_m})$ converges uniformly to a function which is a linear combination of Eisenstein series and (possibly) a $\frac{1}{4}$-cuspidal eigenfunction.

(i) Suppose by contradiction that $S_\infty \in \mathcal{M}_{g,n}$. Then all the cusps of S_∞ are old cusps. Let $s(S_\infty)$ denote the *systole* of S_∞. Then, for $0 < \epsilon < \frac{s(S_\infty)}{2}$ and for m large enough, we have $S_m^{(0,\epsilon)} \subset \cup_{i=1}^n \mathcal{P}_i(m)$. Therefore, applying Corollary 3.3.8, the assumption $\|\phi_m\|_{S_m^{[\epsilon,\infty)}} \to 0$ implies that $\|\phi_m\| \to 0$. This is a contradiction since each ϕ_m is normalized. Thus $S_\infty \in \partial\mathcal{M}_{g,n}$.

(ii) follows from Theorem 3.3.10.

(iii) Fix some ϵ, $0 < \epsilon < \epsilon_0$. Choose constants $K_m \geq 1$ such that

$$\int_{S_m^{[\epsilon,\infty)}} |K_m \phi_m|^2 d\mu_m = 1.$$

Therefore the sequence (K_m) must diverge to ∞. Now we use the mean value formula of Fay and the gradient bound from Chapter 1 along with Corollary 3.3.9 to obtain that, up to extracting a subsequence, $(\widetilde{K_m \phi_m})$ converges on compacta to a function $\widetilde{\phi_\infty}$. By continuity, $\widetilde{\phi_\infty}$ satisfies

$$\Delta \widetilde{\phi_\infty} + \frac{1}{4} \widetilde{\phi_\infty} = 0.$$

Moreover, $\widetilde{\phi_\infty}$ induces a function ϕ_∞ on S_∞ that satisfies

$$\Delta \phi_\infty + \frac{1}{4} \phi_\infty = 0. \tag{3.34}$$

Using the uniform convergence over compacta we have

$$\int_{S_\infty^{[\epsilon,\infty)}} \phi_\infty^2 d\mu_\infty = \lim_{m\to\infty} \int_{S_m^{[\epsilon,\infty)}} K_m \phi_m^2 d\mu_m = 1.$$

Therefore ϕ_∞ is not the zero function. From Lemma 3.3.2 and Lemma 3.3.5 (3.16) we deduce that ϕ_∞ satisfies *moderate growth* condition [Wo, p. 80] in each cusp. It is known that for any $\lambda \geq \frac{1}{4}$ the space of *moderate growth* λ*-eigenfunctions* of S_∞ is spanned by Eisenstein series and (possibly) λ-cuspidal eigenfunctions (see §3 in [Wo]). In particular, ϕ_∞ is a linear combination of Eisenstein series and (possibly) a cuspidal eigenfunction. This finishes the proof of (iii).□

Chapter 4

Small cuspidal eigenvalues of a hyperbolic surface with finite area

In this chapter we try to understand a conjecture from [O-R] on the number of small cuspidal eigenvalues for a surface in $\mathcal{M}_{g,n}$. We shall try to attack the problem using two tools. First one is the topological properties of nodal sets and nodal domains of small cuspidal eigenfunctions. The second is the result about convergence of small cuspidal eigenpairs on converging family of hyperbolic surfaces that is proved in Chapter 3. Using these tools we show in Theorem 4.1.1 that there exists an open, unbounded subset of $\mathcal{M}_{g,n}$ on which the $(2g-1)$-th cuspidal eigenvalue λ^c_{2g-1} is $> \frac{1}{4}$.

4.1 A conjecture of Otal-Rosas

Cuspidal eigenvalues are not very well understood. For an arbitrary Fuchsian group Γ it is not known whether the number of cuspidal eigenvalues of $\mathbb{H}/\Gamma$ is infinite. In a famous theorem, A. Selberg proved that this is indeed the case for Γ arithmetic. Tempted by this result, Selberg conjectured that this should be the case for any hyperbolic surface of finite area. However, in [P-S], Philips and Sarnak has shown that Selberg's conjecture is very unlikely to be true. We shall not go any further into this topic. Rather we focus only on small cuspidal eigenvalues. In [O-R] Jean-Pierre Otal and Eulalio Rosas formulated the following:

Conjecture (Otal-Rosas [O-R]) *Let $S \in \mathcal{M}_{g,n}$. Then $\lambda^c_{2g-2}(S) > \frac{1}{4}$.*

This conjecture is motivated by the following two results.

Proposition (Huxley [Hu], Otal [O]) *Let S be a finite area hyperbolic surface of genus 0 or 1. Then S does not carry any small cuspidal eigenpair.*

Proposition (Otal [O]) *Let S be a finite area hyperbolic surface. Then the multiplicity of a small cuspidal eigenvalue of S is at most $2g-3$.*

For any $N \in \mathbb{N}$ and $t \in \mathbb{R}_{>0}$ we define the sets

$$\mathcal{C}^t_{g,n}(N) = \{S \in \mathcal{M}_{g,n} : \lambda^c_N(S) > t\}.$$

With this notation the conjecture can be formulated by saying that

$$\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-2) = \mathcal{M}_{g,n}.$$

Our methods can not reach the conjecture. However, we prove the following theorem.

Theorem 4.1.1. $\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$ *is an open, unbounded subset of* $\mathcal{M}_{g,n}$.

First we consider the subset $\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}$ of $\partial\mathcal{M}_{g,n} = \overline{\mathcal{M}_{g,n}} \setminus \mathcal{M}_{g,n}$ that consists of those surfaces on which a unique closed geodesic that divides the surface into two surfaces, one of genus g with one puncture and another one of genus zero with $n+1$ punctures, has been pinched. We prove the following proposition that proves the second assertion of Theorem 4.1.1.

Proposition *There exists a neighborhood* $\mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1})$ *of* $\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}$ *in* $\overline{\mathcal{M}_{g,n}}$ *such that* $\lambda^c_{2g-1}(S) > \frac{1}{4}$ *for each* $S \in \mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1})$ *i.e.*

$$\mathcal{N}(\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}) \subset \mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1).$$

Now we briefly sketch a proof of the proposition. We argue by contradiction. Let (S_m) be a sequence in $\mathcal{M}_{g,n}$ that converges to S_∞ in $\mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1} \subset \partial\mathcal{M}_{g,n}$ such that ${\lambda^c}_{2g-1}(S_m) \leq \frac{1}{4}$. For $1 \leq i \leq 2g-1$ and for each m, we choose a small cuspidal eigenpair (λ^i_m, ϕ^i_m) of S_m such that
(i) $\{\phi^i_m\}_{i=1}^{2g-1}$ is an orthonormal family in $L^2(S_m)$,
(ii) λ^i_m is the i-th eigenvalue of S_m.

For $1 \leq i \leq 2g-1$ let (λ^i_m) converges to λ^i_∞ as $m \to \infty$. By Theorem 3.2.1 there are two possible types of behavior that the sequence (ϕ^i_m) can exhibit. Either, for each $1 \leq i \leq 2g-1$ the sequence (ϕ^i_m) converges to a λ^i_∞-eigenfunction ϕ^i_∞ on S_∞, or for some i the sequence (λ^i_m, ϕ^i_m) satisfies condition (2) in Theorem 3.2.1. However, in our case we have the following lemma:

Lemma 1 *For each* i, $1 \leq i \leq 2g-1$, *up to extracting a subsequence, the sequence* (ϕ^i_m) *converges to a* λ^i_∞*-eigenfunction* ϕ^i_∞ *of* S_∞. *The limit functions* ϕ^i_∞ *and* ϕ^j_∞ *are orthogonal for* $i \neq j$ *i.e.* S_∞ *has at least* $2g-1$ *small eigenvalues. Moreover none of the* ϕ^i_∞ *is residual.*

Then we count the number of small eigenvalues of S_∞ using [O-R] to conclude that at least one of ϕ^i_∞ is nonzero on the component of S_∞ of type $(0, n+1)$. This leads to a contradiction by Huxley [Hu] or [O, Proposition 2].

We complete the proof of Theorem 4.1.1 by proving the following lemma using Theorem 2.

Lemma 2 $\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$ *is an open subset of* $\mathcal{M}_{g,n}$.

4.2 Proof of Proposition

Recall that we argue by contradiction. We assume that there is a sequence $S_m \in \mathcal{M}_{g,n}$ converging to $S_\infty \in \mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1} \subset \partial\mathcal{M}_{g,n}$ such that $\lambda^c_{2g-1}(S_m) \leq \frac{1}{4}$. For $1 \leq i \leq 2g-1$ and for each m we choose small cuspidal eigenpairs (λ^i_m, ϕ^i_m) of S_m such that
(i) $\{\phi^i_m\}_{i=1}^{2g-1}$ is an orthonormal family in $L^2(S_m)$,
(ii) λ^i_m is the i-th eigenvalue of S_m.

Theorem 3.2.1 provides two possible behaviors of the sequence (ϕ^i_m). However in our case we have Lemma 1.

Proof of Lemma 1

By uniform convergence of ϕ^i_m to ϕ^i_∞, we have $\|\phi^i_\infty\| \leq 1$. To prove the first two statements of the lemma it is enough to prove that, for $1 \leq i \leq 2g-1$, $\|\phi^i_\infty\| = 1$ because this will imply that ϕ^i_∞ is not the zero function and that (ϕ^i_m) is uniformly integrable over the thick parts: for any $t > 0$ there exists ϵ such that for all m one has,

$$\|\phi_m\|_{S_m^{[\epsilon,\infty)}} > 1 - t.$$

To prove that, for each $1 \leq i \leq 2g-1$, $\|\phi^i_\infty\| = 1$ we argue by contradiction and assume that for some $1 \leq i \leq 2g-1$, $\|\phi^i_\infty\| = 1-\delta$. To simplify the notation, denote the sequence (λ^i_m, ϕ^i_m) by (λ_m, ϕ_m) and the limit $(\lambda^i_\infty, \phi^i_\infty)$ by $(\lambda_\infty, \phi_\infty)$. By Corollary 3.3.8 the functions ϕ_m are uniformly integrable over the union of cusps of S_m: for any $t > 0$ there exists $\epsilon > 0$ such that for all m one has:

$$\|\phi_m\|_{S^{(0,\epsilon)}_{m,c}} < t. \tag{4.1}$$

Since $S_\infty \in \mathcal{M}_{g,1} \cup \mathcal{M}_{0,n+1}$ there is only one closed geodesic, $\gamma_m \subset S_m$, whose length l_{γ_m} tends to zero. For any $l \leq 1$ and for m large enough such that $l_{\gamma_m} < l$ denote by $\mathcal{C}^l_m \subset S_m$ the collar around γ_m bounded by two equidistant curves of length l. In view of the uniform integrability inside cusps (4.1), there exists $\epsilon_0 > 0$ such that for any $\epsilon \leq \epsilon_0$ there exists $m(\epsilon)$ such that for $m \geq m(\epsilon)$ we have:

$$\|\phi_m\|_{\mathcal{C}^\epsilon_m} > \frac{\delta}{2}. \tag{4.2}$$

Now we distinguish again two cases depending on whether $\lambda_\infty < \frac{1}{4}$ or $\lambda_\infty = \frac{1}{4}$. If $\lambda_\infty < \frac{1}{4}$ then we have a contradiction since $\|\phi_\infty\| = 1$ by Theorem 3.3.10. Hence we may suppose that $\lambda_\infty = \frac{1}{4}$. So, by theorem 3.2.1 either ϕ_∞ is the zero function or, for instance by [I, Theorem 3.2], ϕ_∞ is cuspidal. Now recall that by lemma 3.3.5 we have uniform integrability of $[\phi_m]_1$: for any t there exists ϵ such that for all m:

$$\|[\phi_m]_1\|_{\mathcal{C}^\epsilon_m} < t.$$

Hence by (4.2), there exists ϵ_1 such that for any $\epsilon \leq \epsilon_1$ the exists $m_1(\epsilon)$ such that for $m \geq m_1(\epsilon)$ one has:

$$\|[\phi_m]_0\|_{\mathcal{C}^\epsilon_m} > \frac{\delta}{4} \tag{4.3}$$

In particular, if $c(\epsilon, m) = \sup_{z \in \mathcal{C}^\epsilon_m} |[\phi_m]_0|$ then, since area of $\mathcal{C}^\epsilon_m$ is less than 1, we have for any $\epsilon \leq \epsilon_1$ and $m \geq m_1(\epsilon)$:

$$c(\epsilon, m) > \frac{\delta}{4}. \tag{4.4}$$

Now we prove that $[\phi_m]_1$ is uniformly small inside $\mathcal{C}^\epsilon_m$. More precisely,

Lemma 4.2.1. *Let ϵ be such that $0 < \epsilon < 1$. There exists a constant $K < \infty$, independent of ϵ, and $m_2(\epsilon) \in \mathbb{N}$ such that for $m \geq m_2(\epsilon)$ and $z \in \mathcal{C}^\epsilon_m$:*

$$|[\phi_m]_1|(z) < K\frac{\epsilon^{\frac{1}{2}}}{1-\epsilon}.$$

Proof. Consider the expansion of ϕ_m inside $\mathcal{C}^1_m$ with respect to the Fermi coordinates (see **??**):

$$\phi_m(r,\theta) = a^m_0(r) + \sum_{j=1}^{\infty}\Big(a^m_j(r)\cos j\theta + b^m_j(r)\sin j\theta\Big). \tag{4.5}$$

Here, for each $j \geq 0$, (a^m_j, b^m_j) are the j-th Fourier coefficients of ϕ_m inside $\mathcal{C}^1_m$ and are defined for all $|r| \leq L_{1,m}$. Recall that, for any $\epsilon \in [l_{\gamma_m}, 1]$ we denote by $L_{\epsilon,m}$ the number $\cosh^{-1}(\frac{\epsilon}{l_{\gamma_m}})$. Recall also that since ϕ_m is a λ_m-eigenfunction, a^m_j and b^m_j satisfy (3.15) with $2\pi l = l_{\gamma_m}$ and $\lambda = \lambda_m$. Therefore, for $j \geq 1$, one can express:

$$(1)\ a^m_j(r) = a_{m,j}s_{m,j}(r) + b_{m,j}c_{m,j}(r)$$

$$(2)\ b^m_j(r) = a_{m,j}{}'s_{m,j}(r) + b_{m,j}{}'c_{m,j}(r) \tag{4.6}$$

where $s_{m,j}(r)$ and $c_{m,j}(r)$ are the two linearly independent solutions of (3.15) with $l = l(\gamma_m)$ and $\lambda = \lambda_m$.

Recall that $s_{m,j}(r)\cosh^{\frac{1}{2}}(r)$ and $c_{m,j}(r)\cosh^{\frac{1}{2}}(r)$ satisfy:

$$\frac{d^2u}{dr^2} = \left(\frac{1}{4\cosh^2 r} + \frac{j^2}{l^2\cosh^2 r}\right)u.$$

Since, for $r \leq L_{\epsilon,m}$, $l^2\cosh^2 r \leq 1$ by Claim 3.3.6, for each $j \geq 1$, there exists strictly increasing functions $h_{m,j} : [0, L_{1,m}] \to \mathbb{R}_{>0}$ and $k_{m,j} : [0, L_{1,m}] \to \mathbb{R}_{>0}$ such that

$$(i)\ s_{m,j}(r)\sqrt{\cosh(r)} = h_{m,j}(r)\cosh jr$$
$$(ii)\ c_{m,j}(r)\sqrt{\cosh(r)} = k_{m,j}(r)\cosh jr. \tag{4.7}$$

We denote by $\mathcal{P}_{n+1}(\infty)$ and $\mathcal{P}_{n+2}(\infty)$ the two new cusps of S_∞ that appear as the limit of $\mathcal{C}^1_m$ as $m \to \infty$. Now, let us assume:

$$\sup_{z\in\partial\mathcal{P}_{n+1}(\infty)\cup\partial\mathcal{P}_{n+2}(\infty)}|\phi_\infty|(z) < \frac{t}{4}.$$

Then, by the uniform convergence of ϕ_m to ϕ_∞ over compacta, we have a $N \in \mathbb{N}$ such that for $m \geq N$ and $z \in \partial\mathcal{C}^1_m$:

$$|\phi_m|(z) < \frac{t}{4}.$$

By (4.5) for any $j \geq 1$:

$$|a^m_j|(\pm L_{1,m}) = \frac{1}{\pi}\left|\int_0^{2\pi}\phi_m(\pm L_{1,m},\theta)\cos j\theta d\theta\right| \leq \frac{t}{2}. \tag{4.8}$$

Similar calculations for b^m_j provide: $|b^m_j|(\pm L_{1,m}) \leq \frac{t}{2}$. Recall that $s_{m,j}$ is odd and $c_{m,j}$ is even. So by (4.6) and (4.7):

$$(i)\ a^m_j(L_{1,m}) + a^m_j(-L_{1,m}) = 2b_{m,j}k_j(L_{1,m})\frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}}$$
$$(ii)\ a^m_j(L_{1,m}) - a^m_j(-L_{1,m}) = 2a_{m,j}h_j(L_{1,m})\frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}}. \tag{4.9}$$

Therefore, by (4.8) and (4.9):

$$(i)\ |b_{m,j}|k_j(L_{1,m})\frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}} < \frac{t}{2}$$
$$(ii)\ |a_{m,j}|h_j(L_{1,m})\frac{\cosh jL_{1,m}}{\sqrt{\cosh L_{1,m}}} < \frac{t}{2}. \tag{4.10}$$

Therefore, for any $r \leq L_{1,m}$:

$$|a^m_j|(r) = |a_{m,j}s_{m,j}(r) + b_{m,j}c_{m,j}(r)| < |a_{m,j}|s_{m,j}(r) + |b_{m,j}|c_{m,j}(r).$$

The last term of the inequality is

$$|a_{m,j}|h_{m,j}(r)\frac{\cosh jr}{\sqrt{\cosh r}} + |b_{m,j}|k_{m,j}(r)\frac{\cosh jr}{\sqrt{\cosh r}} < t\frac{\cosh jr}{\sqrt{\cosh r}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}}$$

since $h_{m,j}$ and $k_{m,j}$ are strictly increasing functions (by (4.10)). Similarly,

$$|b_j^m|(r) < t\frac{\cosh jr}{\sqrt{\cosh r}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}}.$$

Hence

$$|[\phi_m]_1|(r,\theta) < 2t\sum_{j=1}^{\infty}\frac{\cosh jr}{\sqrt{\cosh r}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}}. \tag{4.11}$$

Since, for $j \geq 1$, the function $\frac{\cosh jr}{\sqrt{\cosh r}}$ is strictly increasing, for any $r \leq L_{\epsilon,m}$:

$$\sum_{j=1}^{\infty}\frac{\cosh jr}{\sqrt{\cosh r}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}} < \sum_{j=1}^{\infty}\frac{\cosh jL_{\epsilon,m}}{\sqrt{\cosh L_{\epsilon,m}}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}} \tag{4.12}$$

Now fix an ϵ such that $0 < \epsilon < 1$. Observe that $L_{\epsilon,m} = \log(\frac{\epsilon}{l_{\gamma_m}} + \sqrt{\frac{\epsilon^2}{l_{\gamma_m}^2} - 1})$. So, for m large such that l_{γ_m} is small compared to ϵ:

$$\sum_{j=1}^{\infty}\frac{\cosh jL_{\epsilon,m}}{\sqrt{\cosh L_{\epsilon,m}}}\frac{\sqrt{\cosh L_{1,m}}}{\cosh jL_{1,m}} < K'\sum_{j=1}^{\infty}\epsilon^j\epsilon^{-\frac{1}{2}} = K'\frac{\epsilon^{\frac{1}{2}}}{1-\epsilon} \tag{4.13}$$

where the constant K' can be chosen independently of ϵ as soon as m is larger than some number $m_2(\epsilon) \in \mathbb{N}$. Therefore, by (4.11) and (4.13), for $m \geq m_2(\epsilon)$ and $(r,\theta) \in \mathcal{C}_m^\epsilon$

$$|[\phi_m]_1|(r,\theta) < 2tK'\frac{\epsilon^{\frac{1}{2}}}{1-\epsilon}. \tag{4.14}$$

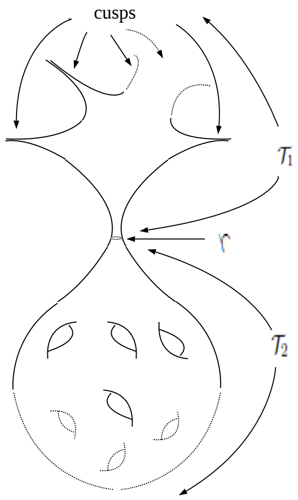
This proves the lemma.□

Now fix $\epsilon < \epsilon_1$ (see (4.3)) such that $K\frac{\epsilon^{\frac{1}{2}}}{1-\epsilon} < \frac{\delta}{4}$ and choose $m \geq \max\{m_1(\epsilon), m_2(\epsilon)\}$. Then by Lemma 4.2.1 and (4.4): for each $z \in \mathcal{C}_m^\epsilon$

$$c(\epsilon, m) > |[\phi_m]_1|(z). \tag{4.15}$$

So the parallel curve α_m with distance r_0 ($\leq L_{\epsilon,m}$) from γ_m such that $c = |[\phi_m]_0|(r_0)$ has the property that ϕ_m has constant sign on it. In other words, the nodal set $\mathcal{Z}(\phi_m)$ does not intersect this curve. This is a contradiction to the next lemma.

Lemma 4.2.2. *Let S be a noncompact, finite area hyperbolic surface of type (g,n). Let γ be a simple closed geodesic that separates S into two connected components $\mathcal{T}_1$ and $\mathcal{T}_2$ such that $\mathcal{T}_1$ is topologically a sphere with $n+1$ punctures and $\mathcal{T}_2$ is topologically a genus g surface with one puncture. Let f be a small cuspidal eigenfunction of S. Then the zero set $\mathcal{Z}(f)$ of f intersects every curve homotopic to γ.*



Proof. Recall that $\mathcal{Z}(f)$ is a locally finite graph [Ch]. Let us assume that $\mathcal{Z}(f)$ does not intersect some curve τ homotopic to γ. We have $S \setminus \tau = \mathcal{T}_1 \cup \mathcal{T}_2$ and all the punctures of S are contained in $\mathcal{T}_1$. Consider the components of $\mathcal{T}_1 \setminus \mathcal{Z}(f)$. Recall that since f is cuspidal $\mathcal{Z}(f)$ contains all the punctures of S and therefore these components give rise to a cell decomposition of a once punctured sphere. The Euler characteristic of the component $\mathcal{F}$ containing τ as a puncture is either negative or zero (since γ and each component of $\mathcal{Z}(f)$ are essential; see [O]). Each component of $\mathcal{T}_1 \setminus \mathcal{Z}(f)$ other than $\mathcal{F}$ (at least one such exists since g changes sign in $\mathcal{T}_1$) is a *nodal domain* of f and hence has negative Euler characteristic [O]. Also $\mathcal{Z}(f)$ being a graph has non-positive Euler characteristic. Let C^+ (resp. C^-) be the union of the nodal domains contained in $\mathcal{T}_1$ which are different from $\mathcal{F}$ and where f is positive (resp. negative). Denote by $\chi(X)$ the Euler characteristic of the topological space X. Since the Euler characteristic of a once punctured sphere is 1, by the Euler-Poincaré formula one has:

$$1 = \chi(\mathcal{F}) + \chi(C^+) + \chi(C^-) + \chi(\mathcal{Z}(f)).$$

This is a contradiction because the right hand side of the equality is strictly negative.□

Now we prove that ϕ_∞ is not a residual eigenfunction. It is clear from the uniform convergence that ϕ_∞ is cuspidal at the old cusps. If ϕ_∞ is a residual eigenfunction then the only possibility is that ϕ_∞ is not cupsidal at one of the two new cusps. Let us assume that ϕ_∞ is residual in $\mathcal{P}_{n+1}$. Then, for sufficiently large t, ϕ_∞ has constant sign in $\mathcal{P}^t_{n+1}$. Therefore, by the uniform convergence $\phi_m|_{S_m^{[\epsilon,\infty)}} \to \phi_\infty|_{S_\infty^{[\epsilon,\infty)}}$ it follows that, for all m large, ϕ_m has constant sign on a component of $\partial\mathcal{C}_m^{\frac{1}{t}}$. Since this component is homotopic to γ_m this leads to a contradiction to Lemma 4.2.2 as well. This finishes the proof of Lemma 1.□

Continuation of Proof of Proposition

Let us denote the two components of S_∞ by $\mathcal{N}_1$ and $\mathcal{N}_2$ such that $\mathcal{N}_1 \in \mathcal{M}_{g,1}$ and $\mathcal{N}_2 \in \mathcal{M}_{0,n+1}$. Lemma 1 says that S_∞ must have at least $2g-1$ many small cuspidal eigenvalues. By [O-R, Théoréme 0.2] the number of non-zero small eigenvalues of $\mathcal{N}_1$ is at most $2g-2$. In particular, the number of small cuspidal eigenvalues of $\mathcal{N}_1$ is at most $2g-2$. Thus for some i, $1 \leq i \leq 2g-1$, ϕ^i_∞ is not the zero function when restricted to $\mathcal{N}_2$ i.e. ϕ^i_∞ is a cuspidal eigenfunction of $\mathcal{N}_2$. This is a contradiction because $\mathcal{N}_2$ does not have any small cuspidal eigenfunction by [H] or [O]. □

Remark 4.2.3. *The arguments in the proof of Proposition are applicable to more general settings. In particular, let (S_m) be a sequence in $\mathcal{M}_{g,n}$ that converges to $S_\infty \in \partial\mathcal{M}_{g,n}$. Let (λ_m, ϕ_m) be a normalized small eigenpair of S_m. Let $\lambda_m \to \lambda_\infty$ as m tends to infinity. The arguments show the following: If $\liminf_{m\to\infty}\|\phi_m\| < 1$ then there exists a curve α_m, homotopic to a geodesic of length tending to zero, on which, up to extracting a subsequence, ϕ_m has constant sign.*

4.2.4 Proof of Lemma 2

Now we prove Lemma 2 which says that $\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$ is open in $\mathcal{M}_{g,n}$.

We argue by contradiction and assume that there exists a $S \in \mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$ such that every neighborhood of S contains points from $\mathcal{M}_{g,n}\backslash\mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$. In other word, there exists a sequence $(S_m) \subseteq \mathcal{M}_{g,n}$ that converges to S and, for all m, $\lambda^c_{2g-1}(S_m) \leq \frac{1}{4}$. For $1 \leq i \leq 2g-1$, let us denote by ϕ^i_m a normalized $\lambda^c_i(S_m)$-cuspidal eigenfunction such that $\{\phi^i_m\}_{i=1}^{2g-1}$ is an orthonormal family in $L^2(S_m)$. Since we are considering small eigenvalues, up to extracting a subsequence, the sequence $(\lambda^c_i(S_m))$ converges. For simplicity we assume that, for $1 \leq i \leq 2g-1$, the sequence $(\lambda^c_i(S_m))$ converges and denote by λ^i_∞ its limit. Observe that, for $1 \leq i \leq 2g-1$, $\lambda^i_\infty \leq \frac{1}{4}$. Now, since $S \in \mathcal{M}_{g,n}$ by Theorem 3.2.1, up to extracting a subsequence, (ϕ^i_m) converge to λ^i_∞-eigenfunction ϕ^i_∞ of S. Since S_m converges to S in $\mathcal{M}_{g,n}$, one has: $\|\phi^i_\infty\| = 1$ by the result about uniform integrability inside cusps in Corollary 3.3.8. Hence $\{\phi^i_\infty\}_{i=1}^{2g-1}$ is an orthonormal family in $L^2(S)$ so that the $(2g-1)$-th cuspidal eigenvalue $\lambda^c_{2g-1}(S)$ of S is below $\frac{1}{4}$. This is a contradiction because by our assumption $S \in \mathcal{C}^{\frac{1}{4}}_{g,n}(2g-1)$. □

Chapter 5

Maximization of λ_1 over closed hyperbolic surfaces of fixed genus

In this chapter we consider λ_1 as a function on the moduli space $\mathcal{M}_g$ of closed hyperbolic surfaces of genus g. It is a bounded continuous functio. In this chapter we discuss the question whether λ_1 can take values more than $\frac{1}{4}$ or not. Using similar topological arguments as those used in Chapter 2 we prove that the question has a positive answer in genus two case.

5.1 Each λ_i is bounded

Recall that for a closed hyperbolic surface S the set of eigenvalues, *spectrum*, of S is a discrete set:

$$0 = \lambda_0(S) < \lambda_1(S) \leq \lambda_2(S) \leq ... \leq \lambda_n(S) \leq ...\infty$$

where each number in the above sequence is repeated according to its multiplicity as an eigenvalue and $\lambda_i(S)$ denotes the i-th eigenvalue of S. We review quickly a proof of the classical result that, for each i, λ_i is continuos and bounded. The Theorem of Lizhen Ji, stated in Chapter 3, provides the following:

Theorem 5.1.1. *Let (S_n) be a sequence of surfaces in $\mathcal{M}_g$ that converges to S_∞ which also belongs to $\mathcal{M}_g$. Let (λ_n, ϕ_n) be a normalized eigenpair of S_n and let λ_n tends to $\lambda_\infty < \infty$. Then λ_∞ is an eigenvalue of S_∞. Moreover, there is a normalized λ_∞-eigenfunction ϕ_∞ such that, up to extracting a subsequence, $\phi_n \to \phi_\infty$ uniformly.*

Therefore, for any i, λ_i is a *lower semi-continuous* function on $\mathcal{M}_g$. A technic used in [C-C] can be adapted to obtain the following:

Theorem 5.1.2. *Let S be a closed hyperbolic surface of genus g. Let $S_n \in \mathcal{M}_g$ such that (S_n) converges to S. Let λ_i be the i-th eigenvalue of S. Then, for any $\epsilon > 0$, there exists $m(\epsilon) \in \mathbb{N}$ such that, for $n \geq m(\epsilon)$, each S_n has at least i many eigenvalues below $\lambda_i + \epsilon$.*

Hence λ_i indeed is a continuous function.

Recall that *systole* of a hyperbolic surface S is defined to be the infimum of lengths of closed geodesics on S. A result of Ber's implies that the set $\mathcal{I}_t = \{M \in \mathcal{M}_g : s(M) \geq t\}$ is compact. Hence by continuity: λ_1 is bounded on $\mathcal{I}_t$. On the other hand, a construction of Buser [**?**, p-219] shows that for any chosen $\epsilon > 0$ there is a $\delta > 0$ such that $\lambda_1(S) \leq \frac{1}{4} + \epsilon$ for any $S \in \mathcal{M}_g \setminus \mathcal{I}_\delta$. Therefore λ_1 is bounded. So we may consider the quantity

$$\Lambda_1(g) = \sup_{S \in \mathcal{M}_g} \lambda_1(S).$$

5.2 Surfaces with large λ_1

For a fixed genus g consider the quantity:

$$\Lambda_1(g) = \sup_{S \in \mathcal{M}_g} \lambda_1(S).$$

Arguments in the previous section show that $\Lambda_1(g)$ is finite for all g. In [B1] P. Buser poses the problem of positivity of the limit

$$\Lambda = \overline{\lim_{g \to \infty}} \Lambda_1(g).$$

In [B2], using methods from analytic number theory, he proves that $\Lambda \geq \frac{3}{16}$. Later in [BBD] Buser, M. Burger and J. Dodziuk construct closed hyperbolic surfaces S such that $\lambda_1(S) \geq \frac{1}{4} - \epsilon$ for any small preassigned $\epsilon > 0$. In [B-M] R. Brooks and E. Makover consider the quantity

$$\Lambda_a = \inf_\Gamma \lambda_1(\mathbb{H}/\Gamma)$$

where Γ is an arithmetic subgroup. For any $\epsilon > 0$, they construct a larger class of closed hyperbolic surfaces S such that $\lambda_1(S) \geq \Lambda_a - \epsilon$. It was already observed in [B1] that $\Lambda \leq \frac{1}{4}$.

5.3 Closed hyperbolic surfaces of genus 2

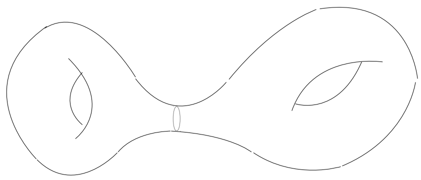
In view of the above results one is tempted to conjecture that $\Lambda \geq \frac{1}{4}$. This will certainly be true if one could prove that $\Lambda_1(g) \geq \frac{1}{4}$. In this section we prove that such an inequality is true when $g = 2$, i.e. we have

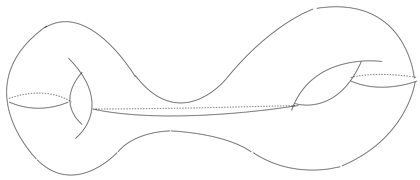
Theorem 5.3.1. $\Lambda_1(2) > \frac{1}{4}$.

Proof. We argue by contradiction and assume that $\Lambda_1(2) \leq \frac{1}{4}$. Therefore, for any $S \in \mathcal{M}_2$: $\lambda_1(S) \leq \frac{1}{4}$. By [O-R] for any $S \in \mathcal{M}_2$: $\lambda_2(S) > \frac{1}{4}$. Hence to a surface $S \in \mathcal{M}_g$ one can assign the first non-constant eigenfunction ϕ_S without any ambiguity. We assume it is normalized. Let $\mathcal{Z}(\phi_S)$ denote the *nodal set* of ϕ_S. Since ϕ_S is the first eigenfunction, by Courant's nodal domain theorem, $S \setminus \mathcal{Z}(\phi_S)$ has exactly two components. As in previous Chapters, we denote by $C^+(\phi_S)$ (resp. $C^-(\phi_S)$) the component of $S \setminus \mathcal{Z}(\phi_S)$ where ϕ_S is positive (resp. negative). From [O, Proof of Proposition 2] we have the following equality:

$$\chi(S) = \chi(C^+(\phi_S)) + \chi(C^-(\phi_S)) + \chi(\mathcal{Z}(\phi_S)). \tag{5.1}$$

Since S has genus 2: $\chi(S) = -2$. Also, by [O, Lemma 1] (see Chapter 1), $\chi(C^+(\phi_S))$ and $\chi(C^-(\phi_S))$ are strictly negative. So (5.1) implies that $\chi(\mathcal{Z}(\phi_S)) = 0$. This means that $\mathcal{Z}(\phi_S)$ consists either of a unique simple closed curve that divides S into two tori with one hole or of tree simple closed curves that divide S into two pair of pants (see below).





Therefore we have the following:

Claim 5.3.2. *Any $S \in \mathcal{M}_g$ has a neighborhood $\mathcal{N}(S)$ such that for any $S^{'} \in \mathcal{N}(S)$ the nodal set $\mathcal{Z}(\phi_{S'})$ is isotopic to $\mathcal{Z}(\phi_S)$.*

Proof. Let $\mathcal{Z}(\phi_S)$ consists of only one curve. Denote the two nodal domains by T_1 and T_2. Let ϕ_S has positive sign on T_1. So necessarily ϕ_S has negative sign on T_2. Now consider a tubular neighborhood $\mathcal{T}_S$ of $\mathcal{Z}(\phi_S)$. By Theorem 5.1.1 we have a neighborhood $\mathcal{N}(S)$ of S such that for any $S^{'} \in \mathcal{N}(S)$, $\phi_{S'}$ has positive sign on $T_1 \setminus \mathcal{T}_S$ and negative sign on $T_2 \setminus \mathcal{T}_S$. Therefore, $\mathcal{Z}(\phi_{S'}) \subset \mathcal{T}_S$. Now the the claim follows from the fact that $\mathcal{Z}(\phi_{S'})$ is incompressible. This same arguments can be applied to the second possibility i.e. when $\mathcal{Z}(\phi_S)$ consists of three curves that divide S into two pair of pants. This finishes the claim. □

Therefore, there exists $S \in \mathcal{M}_2$ such that $\mathcal{Z}(\phi_S)$ consists of only one curve if and only if for all $S^{'} \in \mathcal{M}_2$, $\mathcal{Z}(\phi_{S'})$ consists of only one curve. Now we prove that this is not possible.

Claim 5.3.3. *There exist a surfaces S_1 and S_2 in $\mathcal{M}_2$ such that $\mathcal{Z}(\phi_{S_1})$ consists of only one curve and $\mathcal{Z}(\phi_{S_2})$ consists of three curves.*

Proof. We shall prove existence of a surface $S_1 \in \mathcal{M}_2$ such that $\mathcal{Z}(\phi_{S_1})$ consists of only one curve. Same arguments work to show existence of a surface $S_2 \in \mathcal{M}_2$ such that $\mathcal{Z}(\phi_{S_2})$ consists of three curves.

Consider a sequence of surfaces (S_n) in $\mathcal{M}_2$ that converges to $S_\infty \in \mathcal{M}_{1,1} \cup \mathcal{M}_{1,1} \subset \partial\mathcal{M}_2$. Therefore on S_n the marked geodesic γ_n gets pinched. We also assume that each S_n has an orientation reversing isometry ι which fixes γ_n pointwise. Since the number of components of the limit surface $S_\infty \in \overline{\mathcal{M}_2}$ is exactly two, by [C-C], $\lambda_1(S_n) \to 0$. Now under the action of ι one has two possible behavior for ϕ_{S_n}. If, for some n, ϕ_n is skew invariant under ι then $\gamma_n \subset \mathcal{Z}(\phi_n)$ since γ_n is fixed under ι. Therefore we may assume that ϕ_{S_n} is invariant under ι.

By Lizhen Ji's theorem (see Chapter 3), the sequence ϕ_{S_n}, up to extracting a sequence, converges to the constant function on each component of S_∞ (observe that the limit function is also invariant under the isometry which makes it non-zero on both the components). Thus for any small $\epsilon > 0$ there exists $n(\epsilon) \in \mathbb{N}$ such that $\mathcal{Z}(\phi_{S_n})$ does not intersect $S_n^{[\epsilon,\infty)}$ for $n \geq n(\epsilon)$. Therefore $\mathcal{Z}(\phi_{S_n})$ must be contained inside the *pinching collar* around γ_n for $n \geq n(\epsilon)$. Now observe that $\mathcal{Z}(\phi_{S_n})$ is simple because $\lambda_1(S_n)$ is small. Also, by Courant's nodal domain theorem, it should disconnect the surface. Hence $\mathcal{Z}(\phi_{S_n})$ is isotopic to γ_n for $n \geq n(\epsilon)$. This proves the lemma. □

Next we consider the set

$$\mathcal{B}_g(t) = \{S \in \mathcal{M}_g : \lambda_1(S) > t\}.$$

With this notation Theorem 1 reads as $\mathcal{B}_2(\frac{1}{4}) \neq \emptyset$. Now we prove the following lemma.

Lemma 5.3.4. *$\mathcal{B}_2(\frac{1}{4})$ is an unbounded, open subset of $\mathcal{M}_2$ which disconnects $\mathcal{M}_2$.*

Proof. We have already made the observation that $\mathcal{B}_2(\frac{1}{4})$ is open and non-empty. Now we prove that $\mathcal{B}_2(\frac{1}{4})$ is disconnecting. We argue by contradiction and assume that $\mathcal{M}_2 \setminus \mathcal{B}_2(\frac{1}{4})$ is connected. Observe that, for any $S \in \mathcal{M}_2 \setminus \mathcal{B}_2$, $\lambda_1(S)$ is simple (since $\lambda_2(S) > \frac{1}{4}$). Now we consider two surfaces S_1 and S_2 in $\mathcal{M}_2$ provided by Claim 1 such that $\mathcal{Z}(\phi_{S_1})$ is not isotopic to $\mathcal{Z}(\phi_{S_2})$. As before we connect them by a path η inside $\mathcal{M}_2 \setminus \mathcal{B}_2(\frac{1}{4})$. Using the same arguments as before we must have a surface $S \in \mathcal{M}_2$ such that $\mathcal{Z}(\phi_S)$ is not simple. Hence at least one of the components of $S \setminus \mathcal{Z}(\phi_S)$ has non-negative Euler characteristic. Hence $\lambda_1(S) > \frac{1}{4}$. This contradicts our assumption. Now, $\mathcal{B}_2(\frac{1}{4})$ can not be bounded because there are paths in $\mathcal{M}_g$ that joins S_1 and S_2 and avoids any bounded subset. Hence $\mathcal{B}_2(\frac{1}{4})$ is unbounded. This finishes the lemma. □

Bibliography

[B] Bers, L.; Spaces of degenerating Riemann surfaces, Ann. of Math. Studies 79 (1974), 43-55.

[BJS] Bers, L.; John, F.; Schecter, M.; Partial Differential Equations. New York: Inter-science 1964.

[Bu] Buser, Peter; Geometry and spectra of compact Riemann surfaces. Progress in Mathematics, 106. Birkhäuser Boston, Inc., Boston, MA, 1992.

[B1] Buser, Peter; Cubic graphs and the first eigenvalue of a Riemann surface. Math. Z. 162 (1978), 87-99.

[B2] Buser, Peter; On the bipartition of graphs, Discrete Applied Mathematics, 9 (1984), 105-109.

[BBD] Burger, M., Buser, P., Dodziuk, J., Riemann surfaces of large genus and large λ_1. Geometry and Analysis on Manifolds (T. Sunada, ed.), Lecture Notes in Math. 1339, Springer-Verlag, Berlin, 1988, 54-63.

[B-C] Buser, Peter; Courtois, Gilles; Finite parts of the spectrum of a Riemann surface. Math. Ann. 287, 523 - 530 (1990)

[BJS] Bers, L. John, F., Schecter, M.; Partial Differential Equations. New York: Inter-science 1964.

[B-M] Brooks, R., Makover E., Riemann surfaces with large first eigenvalue. J. Anal. Math. 83 (2001), 243 - 258

[B-Z] Burago, Yu. D.; Zalgaller, V. A. Geometric inequalities. Translated from the Russian by A. B. Sosinskiĭ. Grundlehren der Mathematischen Wissenschaften, 285. Springer Series in Soviet Mathematics. Springer-Verlag, Berlin.

[C-C] Colbois, Bruno; Courtois, Gilles; Les valeurs propres inférieures *à* 1/4 des surfaces de Riemann de petit rayon d'injectivité. Comment. Math. Helv. 64 (1989), no. 3, 349 - 362.

[Cha] Chavel, Isaac; Eigenvalues in Riemannian geometry. Pure and Applied Mathematics, 115. Academic Press, 1984.

[Ch] Cheng, Y., Eigenfunctions and nodal domains. Comment. Math. Helv. 51 (1976), 43-55

[D-P-R-S] Dodziuk J., Pignataro, Randol B., Sullivan D., Estimating small eigenvalues of Riemann surfaces, Contemp. Math. vol 64, Amer. Math. Soc. Prevedence, RI, 1987, pp 93-121.

[F] Fay, J. D.; Fourier coefficient of the resolvent for a Fuchsian group. J. Reine Angew. Math. 293, 143-203 (1977).

[G-T] Gilbarg, David; Trudinger, Neil S.; Elliptic partial differential equations of second order. Reprint of the 1998 edition. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[H] Henrot, Antoine; Extremum problems for eigenvalues of elliptic operators. Frontiers in Mathematics. Birkhäuser Verlag, Basel, 2006.

[He] Hejhal, D. ; Regular b-groups, degenerating Riemann surfaces and spectral theory, Memoires of Amer. Math. Soc. 88, No. 437, 1990.

[Hu] Huxley, M. N.; Cheeger's inequality with a boundary term, Commentarii Mathematici Helvetici 58 (1983).

[I] Iwaniec, H., Introduction to the Spectral Theory of Automorphic Forms, Bibl. Rev. Mat. Iberoamericana, Revista Matemática Iberoamericana, Madrid, 1995.

[J] Judge, C.; Tracking eigenvalues to the frontier of moduli space. I; J. Funct. Anal. 184 (2001), no. 2, 273- 290.

[Ji] Ji, Lizhen; Spectral degeneration of hyperbolic Riemann surfaces. J. Differential Geom. 38 (1993), no. 2, 263 - 313.

[K] Keen, L.; Collars on Riemann surfaces, Discontinuous Groups and Riemann Surfaces, Ann. of Math. Studies No. 79, Princeton University Press, Princeton, NJ, 1974, 263 - 268.

[Le] Lebedev, N. N., Special Functions and their Applications. Dover Publications, New York, 1972.

[O] Otal, Jean-Pierre, Three topological properties of small eigenfunctions on hyperbolic surfaces. Geometry and Dynamics of Groups and Spaces, Progr. Math. 265, Birkhäuser, Bassel, 2008.

[O-R] Otal, Jean-Pierre; Rosas, Eulalio; Pour toute surface hyperbolique de genre g, $\lambda_{2g-2} > 1/4$. Duke Math. J. 150 (2009), no. 1, 101 - 115.

[P-S] Philip, R. S. , Sarnak, P.; On cusps forms for co-finite subgroups of $PSL(2, \mathbb{R})$, Invent. Math. 80 (1985), 339 - 364.

[R1] Randol, B.; Small eigenvalues of the Laplace operator on compact Riemann surfaces, Bull. Amer. Math. Soc. 80 (1974), 996-1000.

[R2] Randol, B.; Cylinders in Riemann surfaces, Comm. Math. Helv. 54, 1979, pp. 1-5.

[R3] Randol, B.; A remark on λ_{2g-2}, Proc. Amer. Math. Soc. 108 (1990), 1081-1083.

[Sch] Schmutz, P.; Small eigenvalues on Riemann surfaces of genus 2, Invent. Math. 106 (1991), 121-138.

[Se] Sévennec, Bruno; Multiplicity of the second Schrödinger eigenvalue on closed surfaces. Math. Ann. 324 (2002), no. 1, 195 - 211.

[S-W-Y] Schoen, R.; Wolpert, S.; Yau, S. T.; Geometric bounds on the low eigenvalues of a compact surface. Geometry of the Laplace operator (Proc. Sympos. Pure Math., Univ. Hawaii, Honolulu, Hawaii, 1979), pp. 279 - 285.

[W] Wolf, M.; Infinite energy harmonic maps and degeneration of hyperbolic surfaces in Moduli spaces, J. Differential Geometry 33 (1991) 487-539

[Wo] Wolpert, S. A.; Spectral limits for hyperbolic surface, I, Invent. Math. 108 (1992), 67 - 89